Polyphase Representation of a 2-channel F.B.

Suppose $H_0(z) = E_0(z^2) + z^{-1} E_1(z^2)$

If we assume quadrature mirror property,

$H_1(z) = H_0(-z)$

$H_1(z) = E_0(z^2) - z^{-1} E_1(z^2)$

$$\begin{bmatrix} H_0(z) \\ H_1(z) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} E_0(z^2) \\ z^{-1} E_1(z^2) \end{bmatrix} \quad \left(\begin{array}{l} \text{Analysis Bank} \\ 2\phi \text{ representation} \end{array} \right)$$
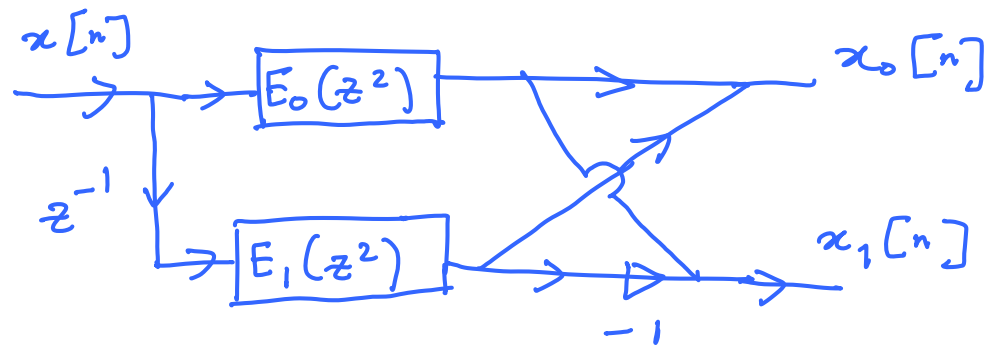
Using alias cancelation conditions,

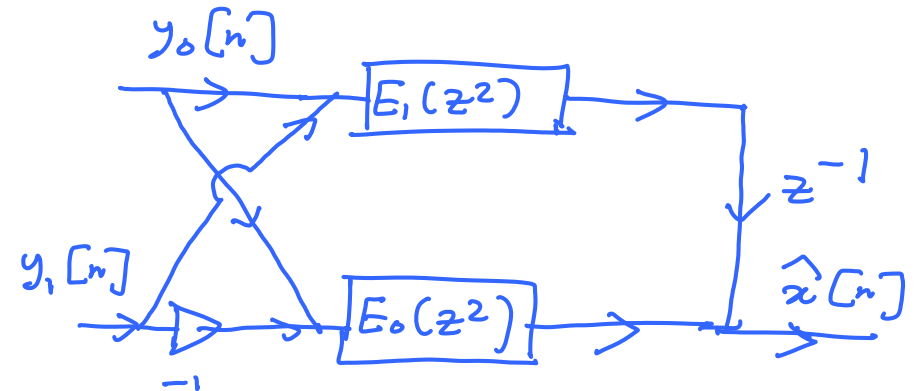$$F_0(z) = H_1(-z) \quad ; \quad F_1(z) = -H_0(-z)$$

$$\begin{bmatrix} F_0(z) & F_1(z) \end{bmatrix} = \begin{bmatrix} z^{-1} E_1(z^2) & E_0(z^2) \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$\left(\text{Synthesis filters realized using } 2\phi \text{ representation} \right)$$

Signal Flow Graphs / Representations for poly ϕ realization

Analysis Bank

$x[n]$ → $E_0(z^2)$, z^{-1} → $E_1(z^2)$, crossover with -1 → $x_0[n]$, $x_1[n]$

Synthesis Bank

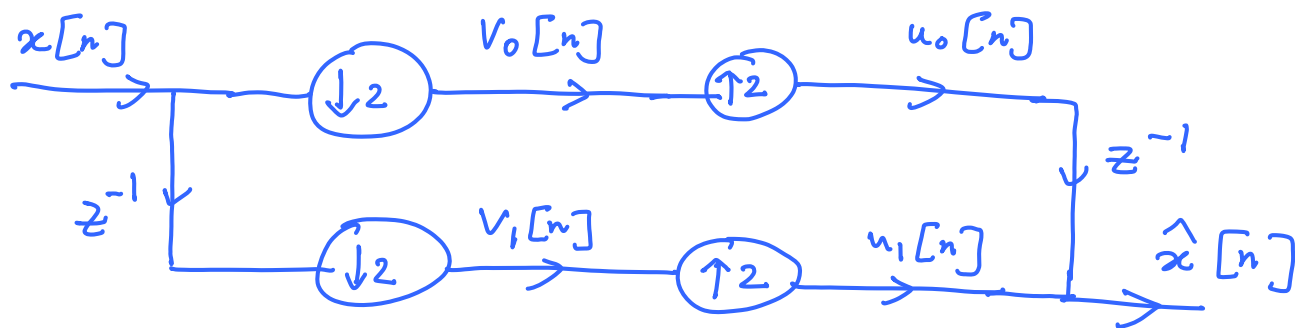$y_0[n]$ → $E_1(z^2)$, $y_1[n]$ → -1 → $E_0(z^2)$, z^{-1} → $\hat{x}[n]$

Example :   2 - channel Perfect reconstruction System

Suppose $H_0(z) = 1$ $H_1(z) = z^{-1}$

$F_0(z) = z^{-1}$ $F_1(z) = 1$

GOAL: Examine if the system in this example is a P. R. system

Suppose

$$x[n] = \{\underset{\underset{n=0}{\uparrow}}{1} \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8 \dots\}$$

$$v_0[n] = \{1 \quad 3 \quad 5 \quad 7 \quad \dots \}$$

$$v_1[n] = \{x \quad 2 \quad 4 \quad 6 \quad 8 \quad \dots \}$$

$\underline{x}$: dummy

$$u_0[n] = \{1 \quad \underline{0} \quad 3 \quad \underline{0} \quad 5 \quad \underline{0} \quad 7 \quad \dots \}$$

$$u_1[n] = \{x \quad \underline{0} \quad 2 \quad \underline{0} \quad 4 \quad \underline{0} \quad \dots \}$$

$$u_0[n-1] = \{x \quad 1 \quad \underline{0} \quad 3 \quad \underline{0} \quad 5 \quad \underline{0} \quad 7 \quad \dots \}$$

$$u_1[n] + u_0[n-1] = \{\underset{\underset{n=0}{\sim}}{x} \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad \dots \}$$

M — channel Filter Banks (GENERALIZATION)

$x[n]$ $x_0[n]$ $v_0[n]$ $u_0[n]$

$H_0(z)$ $\downarrow M$ $\uparrow M$ $F_0(z)$

$H_1(z)$ $\downarrow M$ $\uparrow M$ $F_1(z)$

$H_{M-1}(z)$ $\downarrow M$ $\uparrow M$ $F_{M-1}(z)$ $\hat{x}[n]$

Analysis Bank | Downsamplers   Upsamplers   Synthesis filters
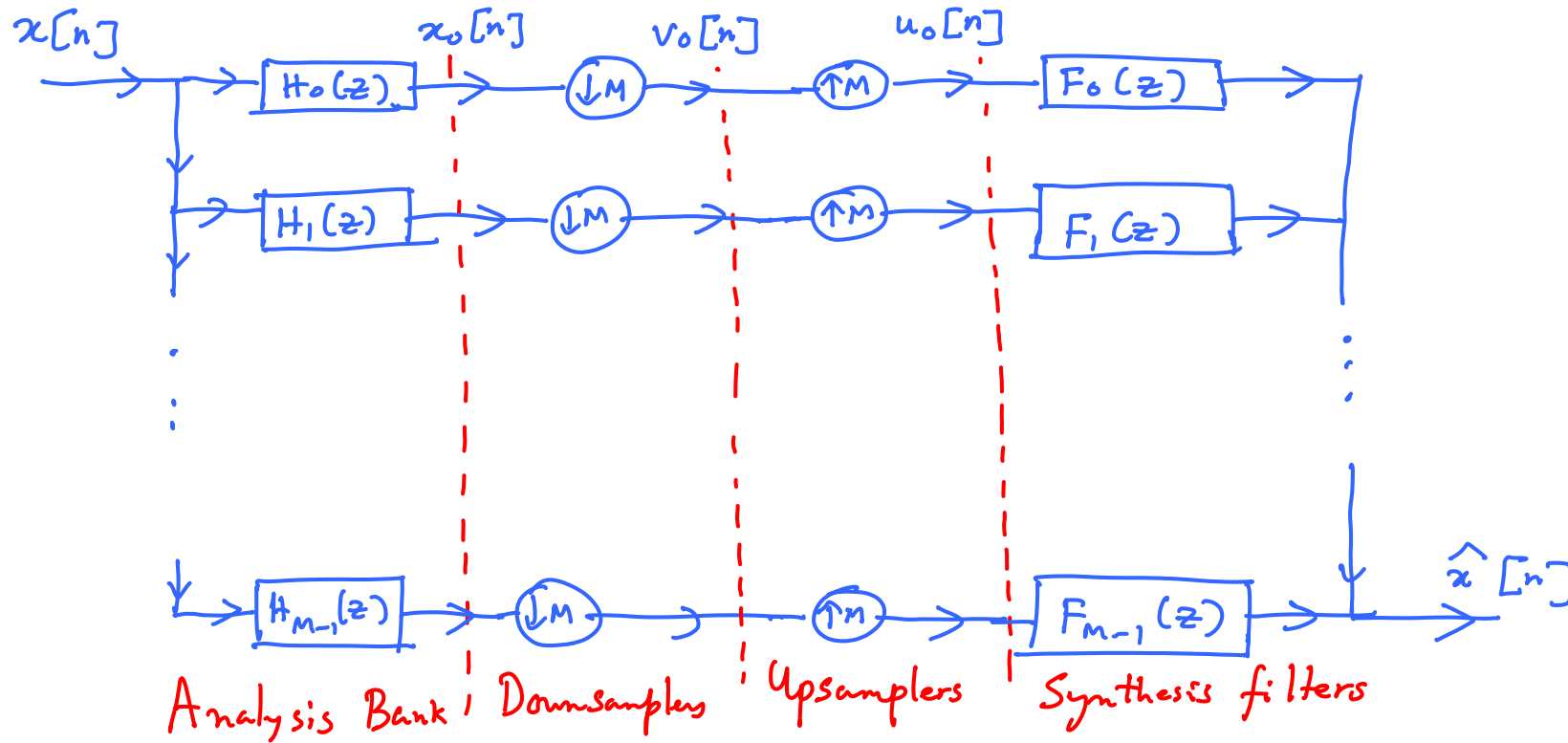
$$\text{let} \quad \underline{h}(z) = \begin{bmatrix} H_0(z) \\ H_1(z) \\ \vdots \\ H_{M-1}(z) \end{bmatrix} \qquad \underline{f}(z) = \begin{bmatrix} F_0(z) \\ F_1(z) \\ \vdots \\ F_{M-1}(z) \end{bmatrix}$$

Analysis Bank

transposed synthesis filters

$$\underline{e}(z) = \begin{bmatrix} 1 \\ z^{-1} \\ \vdots \\ z^{-(M-1)} \end{bmatrix}$$

delay chain needed for poly phase representation

$$X_k(z) = H_k(z) X(z) \qquad k = 0, 1, \ldots, M-1$$

$$V_k(z) = \frac{1}{M} \sum_{l=0}^{M-1} X\left(z^{\frac{1}{M}} \omega^l\right) H_k\left(z^{\frac{1}{M}} \omega^l\right)$$

$$\omega = e^{-j\frac{2\pi}{M}}$$

$$U_k(z) = V_k(z^M) = \frac{1}{M} \sum_{l=0}^{M-1} X\left(z \omega^l\right) H_k\left(z \omega^l\right)$$

$$\hat{X}(z) = \sum_{k=0}^{M-1} F_k(z) U_k(z) = \frac{1}{M} \sum_{l=0}^{M-1} X\left(z \omega^l\right) \sum_{k=0}^{M-1} H_k\left(z \omega^l\right) F_k(z)$$

Rewriting in an easier way,

$$\hat{X}(z) = \sum_{l=0}^{M-1} A_l(z) \, X(z \, w^l) \qquad 0 \le l \le M-1$$

where $A_l(z) = \dfrac{1}{M} \sum_{k=0}^{M-1} H_k(z \, w^l) \, F_k(z)$

With $z = e^{j\omega}$

$$X(e^{j\omega} \, w^l) = X\left(e^{j\left(\omega - \frac{2\pi l}{M}\right)}\right)$$

$X(z \, w^l)$ for $1 \le l \le M-1$ are "<u>ALIAS COMPONENTS</u>"

To avoid aliasing $\qquad A_\ell(z) = 0 \qquad 1 \leq \ell \leq M-1$

$$\text{Let} \quad \underline{A}(z) = \begin{bmatrix} A_0(z) \\ A_1(z) \\ \vdots \\ A_{M-1}(z) \end{bmatrix}$$

Let

$$H(z) = \begin{bmatrix} H_0(z) & H_1(z) & \cdots & H_{M-1}(z) \\ H_0(zw) & H_1(zw) & \cdots & H_{M-1}(zw) \\ \vdots & & \ddots & \\ H_0(zw^{M-1}) & H_1(zw^{M-1}) & \cdots & H_{M-1}(zw^{M-1}) \end{bmatrix}_{M \times M}$$

Alias component matrix

With aliasing canceled out, the distortion function

$$T(z) = \frac{1}{M} \sum_{k=0}^{M-1} H_k(z) F_k(z) = A_0(z)$$

Now

$$M \begin{bmatrix} A_0(z) \\ \vdots \\ A_{M-1}(z) \end{bmatrix} = H(z) f(z) = \underline{t}(z)$$

But $A_\ell(z) = 0$ for $1 \le \ell \le M-1$

$$\underline{t}(z) = M \begin{bmatrix} A_0(z) \\ 0 \\ \vdots \\ 0 \end{bmatrix} = M \begin{bmatrix} T(z) \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$\hat{X}(z) = A^T(z) \underline{X}(z)$$

$$= \frac{1}{M} f^T(z) H^T(z) \underline{X}(z)$$

where $\underline{X}(z) = \begin{bmatrix} X(z) \\ X(zw) \\ \vdots \\ X(zw^{(M-1)}) \end{bmatrix}$

Now,

$$f(z) = H^{-1}(z) t(z)$$

$$\left(\begin{array}{l} \text{Realizable if} \\ H^{-1}(z) \text{ exists} \end{array} \right)$$

For P. R.,

$$\underline{t}(z) = c \begin{bmatrix} z^{-n_0} \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Difficulties with AC inversion matrix

a) Unless $|H(z)|$ is $\neq 0$ for every z, we are in trouble towards getting $H^{-1}(z)$

b) $\underline{f}(z) = \dfrac{1}{\det(\mathbf{H}(z))} \operatorname{adj}(\mathbf{H}(z)) \, \underline{t}(z)$

$\longleftarrow$ Can give rise to a denominator poly. which may not be just a 'delay' with a 'gain'.

Even if analysis filters are FIR, Synthesis filters can be IIR

$\Longrightarrow$ Stability can be a 'problem'!

c) If $\det(\mathbf{H}(z))$ has zero on the unit circle i.e., $z = e^{j\omega_0}$

then $\left| T\left(e^{j\omega_0}\right) \right| = 0$

$\implies$ Severe amplitude distortion around ω_0 !
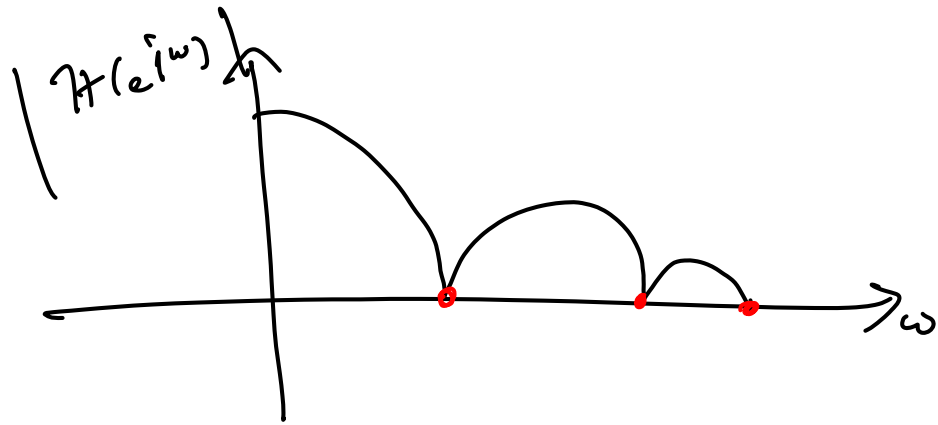
Singularity of $H(e^{j\omega})$ vs. amplitude distortion

If $T(z)$ has a zero @ $z = e^{j\omega_0}$, then

$$t(e^{j\omega_0}) = 0$$

$$\implies \quad \underline{H}(e^{j\omega_0}) \underline{f}(e^{j\omega_0}) = 0$$

$\mathcal{H}(e^{j\omega_0})$ is '$\underline{singular}$' unless all synthesis filters

$F_k(z)$ have a zero @ ω_0

Polyphase Representation for M- channel Filter Banks

From our previous discussion, by polyphase representation,

$$H_k(z) = \sum_{l=0}^{M-1} z^{-l} E_{kl}(z^M) \quad \left(\begin{array}{c} \text{Type 1} \\ \text{representation} \end{array}\right)$$

for analysis filters

We may write

$\underline{h}(z)$

$$\begin{bmatrix} H_0(z) \\ \vdots \\ H_{M-1}(z) \end{bmatrix} = \begin{bmatrix} E_{00}(z^M) & E_{01}(z^M) & \cdots & E_{0,M-1}(z^M) \\ \vdots & \ddots & & \vdots \\ E_{M-1,0}(z^M) & \cdots & & E_{M-1,M-1}(z^M) \end{bmatrix} \begin{bmatrix} 1 \\ z^{-1} \\ \vdots \\ z^{-(M-1)} \end{bmatrix}$$

$\mathbf{E}(z^M)$

$\underline{e}(z)$

$\underline{h}(z) = E(z^M) \cdot \underline{e}(z)$

$$||^{ly}, \text{ we can do this for the synthesis filters}$$

$$F_k(z) = \sum_{l=0}^{M-1} z^{-(M-1-l)} R_{lk}(z^M) \quad \left(\begin{array}{l} \text{Using} \\ \text{Type 2} \\ \text{polyphase} \\ \text{representation} \end{array} \right)$$

$$\underbrace{f^T(z)}_{} \qquad \underbrace{z^{-(M-1)} \cdot}_{\tilde{e}(z)} \qquad \underbrace{R(z^M)}_{}$$

$$\begin{bmatrix} F_0(z) & \cdots & F_{M-1}(z) \end{bmatrix} = \begin{bmatrix} z^{-(M-1)} & \cdots & 1 \end{bmatrix} \begin{bmatrix} R_{00}(z^M) & \cdots & R_{0,M-1}(z^M) \\ \vdots & \ddots & \\ R_{M-1,0}(z^M) & & R_{M-1,M-1}(z^M) \end{bmatrix}$$

$$\underline{f^T(z) = z^{-(M-1)} \, \underline{\tilde{e}}(z) \, R(z^M)} \qquad \qquad \tilde{e}(z) = e(z)^T$$
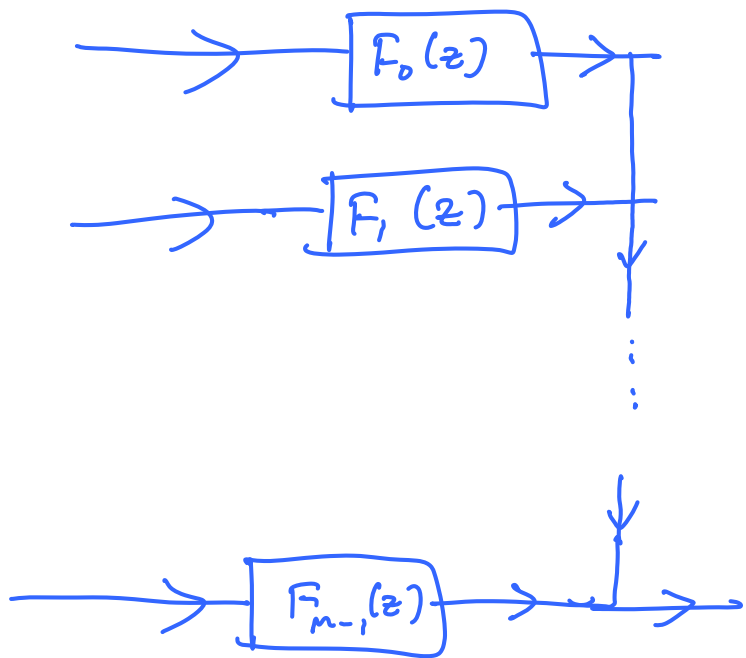
$H_0(z)$

$H_1(z)$

$H_{m-1}(z)$

$\equiv$

z^{-1}
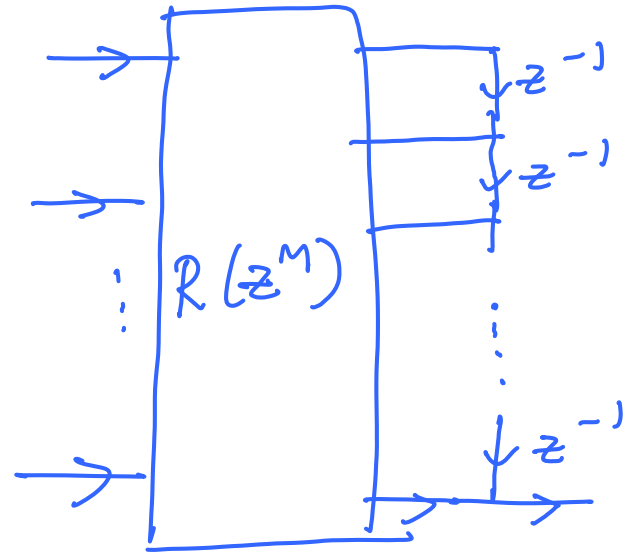
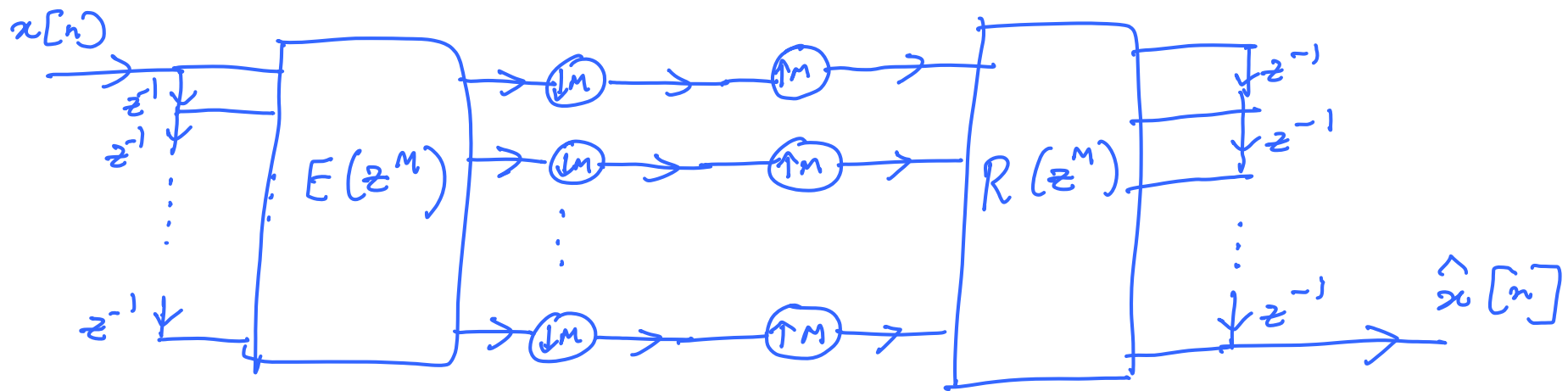z^{-1}

z^{-1}

$E(z^M)$
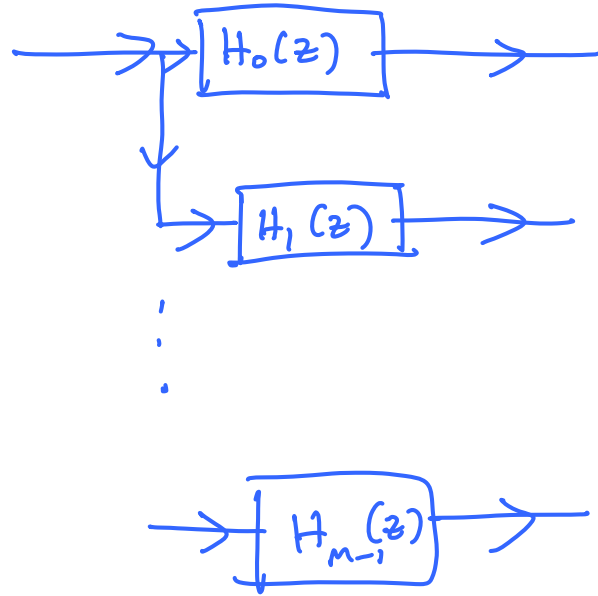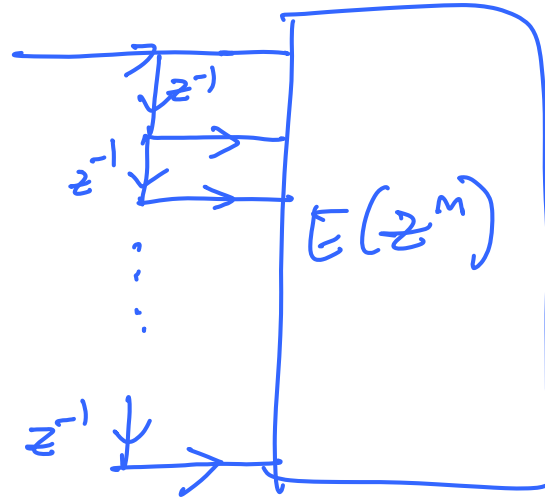
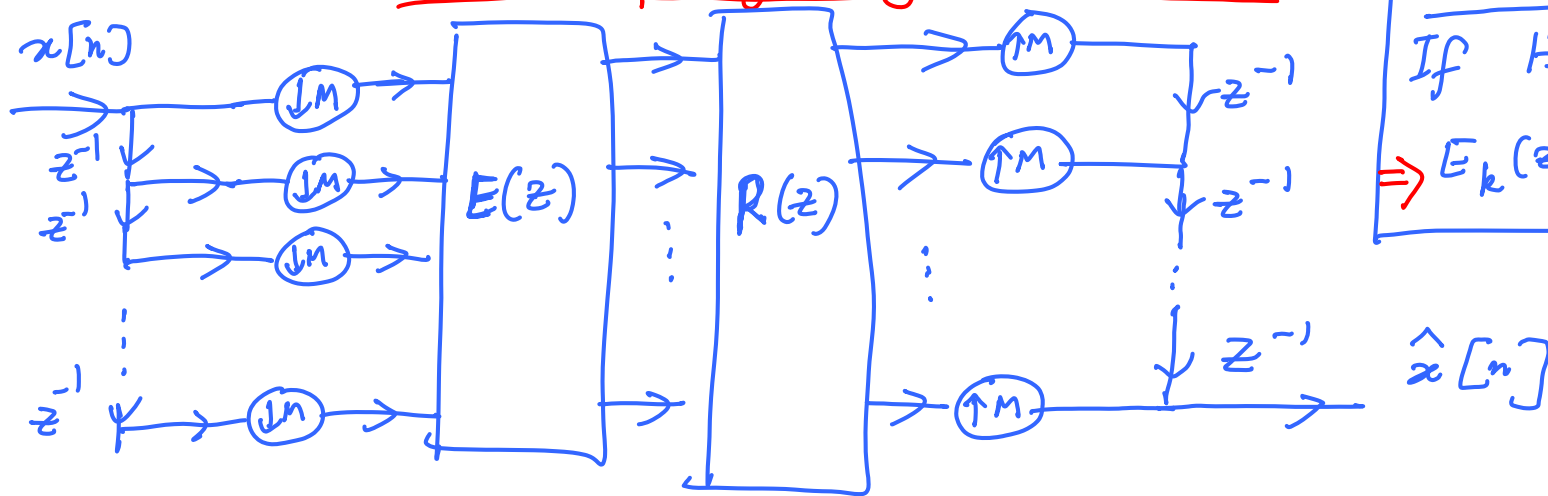Analysis Bank

Synthesis Bank

$R(z^M)$

(Type 2 poly phase)

Architecture - I

Trick: Apply Noble Identities to simplify Arch. I
further towards analysis on P. R. properties

A lot of Symmetry in Arch. II

Aside:
If $H_k(z)$ is Causal
$\Rightarrow E_k(z)$ is Causal

Architecture - II

$$P(z) = E(z) R(z)$$

If $P(z) = I$, we are left with a delay filter bank

GOAL: Choose $R(z)$ to meet the design specs (alias free, P.R.)

From our previous discussion on polyphase decomposition, we saw

$$P(z) = R(z) E(z)$$

If $P(z) = I \implies R(z) = E^{-1}(z)$

Theorem: (TASSP 1987)   (Exercise: Go through the proof)

A necessary and sufficient condition for the P.R. property is $P(z)$ must take one of the following forms i.e.,

$$P(z) = d \begin{bmatrix} z^{-k} & 0 \\ 0 & z^{-k} \end{bmatrix} \quad \text{or} \quad P(z) = d \begin{bmatrix} 0 & z^{-k} \\ z^{-k-1} & 0 \end{bmatrix}$$

$$E(z) = \begin{bmatrix} E_0(z) & E_1(z) \\ E_0(z) & -E_1(z) \end{bmatrix} \qquad R(z) = \begin{bmatrix} E_1(z) & E_1(z) \\ E_0(z) & -E_0(z) \end{bmatrix}$$

$$R(z)\, E(z) = \begin{bmatrix} 2 E_0(z) E_1(z) & 0 \\ 0 & 2 E_0(z) E_1(z) \end{bmatrix}$$

$$= 2 E_0(z) E_1(z) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Now, a system is PR $\underline{\text{iff}}$ $E_0(z) E_1(z)$ is a $\underline{\text{pure delay}}$

If $H_0(z)$ is constrained to be a FIR, it is possible

$\underline{\text{iff}}$ $E_0(z)$ and $E_1(z)$ are pure delays

$\implies$ $H_0(z)$ must be of the form

$$H_0(z) = c_0 z^{-2n_0} + c_1 z^{-(2n_1+1)}$$

If we need a FIR, we may have to give up on

$$H_1(z) = H_0(-z)$$

Idea :

For a general PR, we let $R(z) = d z^{-k} E^{-1}(z)$

a) If $E(z)$ is min ϕ, $R(z)$ hence $\{F_k(z)\}$ will be IIR but 'stable'

b) If some how $\det(E(z))$ lands as a 'delay' we can hopefully construct FIR PR QMF banks

$$\det(E(z)) = z^{-n_0} \implies \text{Family of lossless matrices}$$

This is however not a necessary condition for PR.

Example : Suppose

$$H(z) = \begin{bmatrix} 1 + z^{-1} & 1 - z^{-1} \\ 1 - z^{-1} & 1 + z^{-1} \end{bmatrix}$$

$$H(z)\ \overline{H}(z) = c\,I \qquad (\text{loss less})$$

$$H^{-1}(z) = \frac{1}{4}\begin{bmatrix} 1+z & 1-z \\ 1-z & 1+z \end{bmatrix}$$

Suppose $\quad R(z) = c\,z^{-k}\,\underbrace{E^{-1}(z)}_{\textcircled{I}}$. Consider $E(z) = H(z)$

We plug in $k=1,\ c=4$

$$R(z) = \begin{bmatrix} 1+z^{-1} & z^{-1}-1 \\ z^{-1}-1 & 1+z^{-1} \end{bmatrix}$$

$$E(z) = H(z)$$

$$
\begin{aligned}
H_0(z) &= E_{00}(z^2) + z^{-1} E_{01}(z^2) \\
&= 1 + z^{-2} + z^{-1}(1-z^{-2}) \\
&= 1 + z^{-1} + z^{-2} - z^{-3}
\end{aligned}
$$

$$H_1(z) = E_{10}(z^2) + z^{-1} E_{11}(z^2)$$

$$= 1 - z^{-2} + z^{-1}(1 + z^{-2})$$

$$= 1 + z^{-1} - z^{-2} + z^{-3}$$

III^{ly} for the synthesis bank,

$$F_0(z) = z^{-1} R_{00}(z^2) + R_{10}(z^2)$$

$$F_1(z) = z^{-1} R_{01}(z^2) + R_{11}(z^2)$$

$$F_0(z) = z^{-1}(1 + z^{-2}) + z^{-2} - 1 = -1 + z^{-1} + z^{-2} + z^{-3}$$

$$F_1(z) = 1 - z^{-1} + z^{-2} + z^{-3}$$

Special Filters and Properties

M^{th} band / Nyquist M filters

From polyphase decomposition,

$$H(z) = c + z^{-1} E_1(z^M) + z^{-2} E_2(z^M) + \ldots + z^{-(M-1)} E_{M-1}(z^M)$$

$$x[n] \longrightarrow \boxed{\uparrow M} \longrightarrow \boxed{H(z)} \longrightarrow y[n]$$

$$Y(z) = X(z^M) H(z)$$

$$= c X(z^M) + \sum_{l=1}^{M-1} z^{-l} E_l(z^M) X(z^M)$$

$$y(Mn) = c x(n) \qquad \text{by considering just the 1st polyphase component!}$$
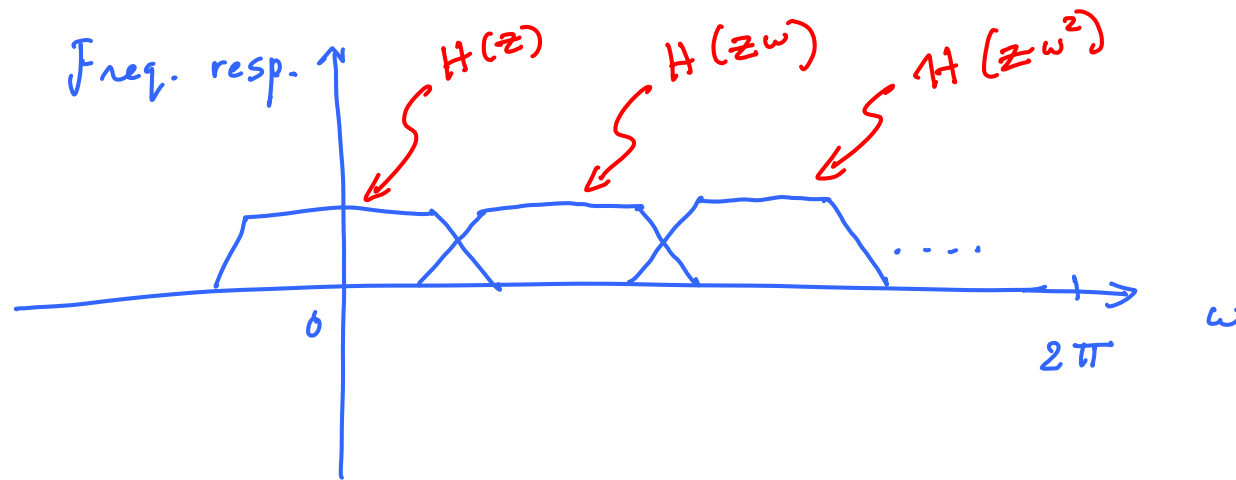
*

Suppose $h(Mn) = \begin{cases} c & n = 0 \\ 0 & \text{else} \end{cases}$

From the representation of $H(z)$ in the polyphase form,

$$\sum_{k=0}^{M-1} H(z \omega^k) = \begin{array}{l} c + z^{-1} E_1(z^M) + \cdots + z^{-(M-1)} E_{M-1}(z^M) \\ + \quad c + z^{-1}\omega^{-1} E_1(z^M \omega^M) + \cdots + z^{-(M-1)}\omega^{-(M-1)} E_{M-1}(z^M \omega^M) \\ \vdots \\ + \quad c + z^{-1}\omega^{-(M-1)} E_1(z^M \omega^{(M-1)M}) + \cdots + z^{-(M-1)}\omega^{-(M-1)(M-1)} \cdot E_{M-1}(z^M \omega^{(M-1)M}) \end{array}$$

$$cM + z^{-1}\underbrace{\left[1 + \omega^{-1} + \cdots + \omega^{-(M-1)}\right]}_{\substack{0 \\ M-1}} E_1(\cdot) + \cdots$$

If $c = \dfrac{1}{M} \implies \displaystyle\sum_{k=0}^{M-1} H(z\omega^k) = 1$

Freq. resp.

H(z)   H(zw)   H(zw²)

0   2π   ω

Ponder: Why are these called `Nyquist' M filters?
What is the cross over frequency for each pair of
adjacent freq. responses?

Half band filters

This is a special case of the Nyquist $M = 2$ filters.

$$H(z) = c + z^{-1} E_1(z^2)$$

$$h(2n) = \begin{cases} c & n = 0 \\ 0 & \text{else} \end{cases}$$

VERIFICATION

$$H(-z) = H(e^{j(\pi + \omega)})$$
$$H(e^{j\omega}) + H(e^{j(\pi + \omega)}) = 1$$
symmetric around $\pi/2$!

Examples: $$H(z) = \begin{cases} 1 + z^{-3} \\ z + 1 + z^{-1} \\ 1 + z^{-1} + z^{-3} \end{cases}$$

$$E_1(z) = z^{-1}$$
$$E_1(z) = 1 + z \quad (\text{anti causal})$$
$$E_1(z) = 1 + z^{-1} \quad (\text{causal})$$

Filters have linear phase, but could be causal / anti causal !

NOTES

1) One can design good half band filters using mirror symmetric properties.

2) The generalization to Nyquist M_r filter design can be carried out through optimization techniques.

System Level Properties

A.

$$\left[H_0(z) \quad H_1(z) \quad \cdots \quad H_{M-1}(z) \right] \text{ are } S.C. \text{ if their } H_0$$

responses add to delay.

$$\sum_{k=0}^{M-1} H_k(z) = c\, z^{-n_0} \qquad c \neq 0$$
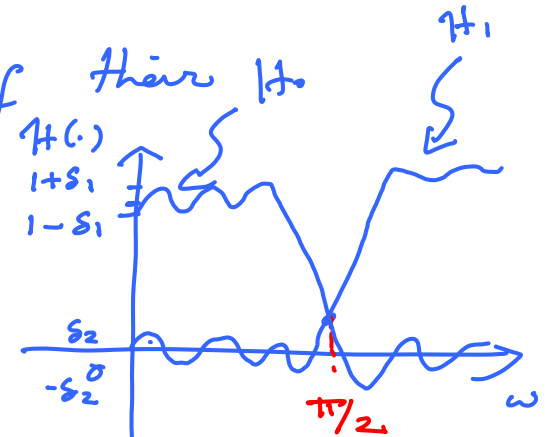
For $M = 2$ case, with $c = 1$ and $n_0 = N/2$,

$$H_0(e^{j\omega}) = e^{-j\omega N/2} H_R(\omega)$$

$$\Rightarrow \quad \begin{array}{l} H_1(z) = z^{-N/2} - H_0(z) \\ H_1(e^{j\omega}) = e^{-j\omega N/2}(1 - H_R(\omega)) \end{array}$$

Fig: Case $M = 2$, $c = 1$, $n_0 = \frac{N}{2}$

B.

$$\underline{\text{Power Complementary}}$$

$$\sum_{k=0}^{M-1} |H_k(e^{j\omega})|^2 = C \qquad \forall \omega$$

$$\Rightarrow \sum_{k=0}^{M-1} H_k(e^{j\omega}) \tilde{H}_k(e^{j\omega}) = C$$

From P. R. perspective, one can choose $F_k(z) = \tilde{H}_k(z)$

To ensure causality, one can impose delay constraints into $\tilde{H}_k(z)$
so that the o/p is delayed

For M = 2, c = 1   PC

$$|H_1(e^{j\omega})|^2 = 1 - |H_0(e^{j\omega})|^2$$

∴ $H_1(e^{j\omega})$ is a spectral factor of $1 - |H_0(e^{j\omega})|^2$

C.

$$\underline{\text{All Pass Complementary}}$$

$$\sum_{k=0}^{M-1} H_k(z) = A(z)$$

where $A(z)$ is an all pass filter

S. C $\Longrightarrow$ A.P.C , but not "$\underline{\text{otherwise}}$"

$\nLeftarrow$

Re construction is free from '$\underline{\text{amplitude distortion}}$'

D.

Doubly Complementary

$\left\{ H_k(z) \right\}_{k=0}^{M-1}$ satisfying both P. C. & A. P. C.

are double complementary