Mathematical Methods and Techniques in Sig. Proc.

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Pre requisites:

1) U.G. course in signals & systems or a basic DSP course.
2) Familiarity in linear algebra / probability & random processes etc.

Signal Proc is an area of applied math dealing with the analysis of signals in discrete / cont. time.

Signal Proc.

Communications (Storage channels, Tx channels such as wireless, optical fibers etc)

Sampling / Transforms
etc

Signal Est. Theory
Sig Det. Theory
⋮

Applications
Speech, Image, Multimedia (audio, video)

Other applications
Biomedical sig. proc.
Quantum Sig. Proc.

Neural Nets. / Pattern Recog.
Non. Linear Sig-Proc.

Mathematical Tools / Techniques used in S.P.

1) Transform Theory
2) Prob. & Stochastic Processes
3) Calculus / Analysis / Functional Analysis
4) Linear Algebra
5) Numerical methods / Approx. Theory.
6) Optimization
7) Stat. Decision Theory (req. (1)-(6))

1. D signals & Systems

Basic Sequences

$$\delta[n] = \begin{cases} 1 & n = 0 \\ 0 & \text{else} \end{cases}$$

$$u[n] = \begin{cases} 1 & n \geq 0 \\ 0 & \text{else} \end{cases}$$

$$x(n) = a^n u(n) \qquad |a| < 1$$

expn. decaying seq.

Systems

$$x[n] \longrightarrow \boxed{T} \longrightarrow y[n]$$

System

Examples

1) Delay system:

$$x(n) \xrightarrow[\text{units}]{\text{Delay by } k} x(n-k)$$

2) M. A (Moving Average)

$$y[n] = \frac{1}{M_1+M_2+1} \sum_{k=-M_1}^{M_2} x(n-k)$$

Memoryless / Memory Systems

O/p $y[n]$ depends only on the i/p $x[n]$

Ex:

$y[n] = x^3[n]$

$y[n] = x^2[n] + 2x[n]$ & so on

Memoryless

$y[n] = x^3[n] + x^2[n-1] + x[n-2]$

Linear / Non linear Systems

All linear systems abide by "superposition" principle

$$T\left(x_1(n) + x_2(n)\right) = T\left(x_1(n)\right) + T\left(x_2(n)\right)$$

$$y_1(n) + y_2(n)$$

Additivity

P1

$$T\left(a\,x(n)\right) = a\,T\left(x(n)\right)$$

(Scaling / Homogenity)

P2

P1 & P2 imply

$$T\left(a_1 x_1(n) + a_2 x_2(n)\right) = a_1 T\left(x_1(n)\right) + a_2 T\left(x_2(n)\right)$$

This is the 'superposition' rule

Time In Variance

Suppose an i/p sequence $x[n]$ is delayed by n_0

$$x_1[n] = x(n-n_0)$$

If the o/p sequence $y(n)$ is '<u>also</u>' delayed by n_0

$$x(n) \Longleftrightarrow y(n)$$

i.e.,
$$y_1(n) = y(n-n_0)$$

then it is called '<u>shift invariance</u>'

<u>Exercise</u> : Suppose $y(n) = x(Ln)$ L is any +ve integer

Is this shift invariant ?

Causality

A system is causal if for every choice of n_0, the o/p sequence @ time $n = n_0$ depends only on the i/p sequence for $n \leq n_0$

Examples :

$$y(n) = \underbrace{x(n+1)}_{\text{sample in the future}} - x(n) \qquad (\text{ANTI CAUSAL!})$$

$$y(n) = x(n) - x(n-1) \qquad (\text{CAUSAL!})$$

'Causality' is a powerful idea!

Stability

A system is (BIBO) bounded i/p bounded o/p stable iff every bounded i/p sequence produces a bounded o/p sequence

I/p is bounded if $|x(n)| \le B_x < \infty \quad \forall n$

BIBO stability requires $|y(n)| \le B_y < \infty \quad \forall n$

Exercise:

$$y(n) = \sum_{k=-\infty}^{n} u(k) = \begin{cases} 0 & n < 0 \\ (n+1) & n \geq 0 \end{cases}$$

Examine if $y(n)$ is bounded?

Answer: Unbounded!

Linear & time invariant systems

Suppose $h_k(n)$ denotes the response of a system to an impulse @ $n = k$ i.e., $\delta(n-k)$

Any sequence $$x(n) = \sum_{k=-\infty}^{\infty} x(k)\,\delta(n-k)$$

Since the system is linear, by superposition,

$$y(n) = T\left(x(n)\right) = T\left(\sum_{k=-\infty}^{\infty} x(k)\,\delta(n-k)\right)$$

$$= \sum_{k=-\infty}^{\infty} x(k)\, T\left\{\delta(n-k)\right\}$$

Linearity

$$= \sum_{k=-\infty}^{\infty} x(k)\, h_k(n)$$

SHIFT INVARIANCE

By shift invariance

Response to $\delta(n-k) \longrightarrow h(n-k)$

i.e., $h_k(n) = h(n-k)$

$$\boxed{y(n) = \sum_{k=-\infty}^{\infty} x(k)\, h(n-k)}$$

CONVOLUTION OPERATOR

Reflection $\longrightarrow$ Shift $\longrightarrow$ Multiply $\longrightarrow$ Add

Modes in a linear system

Often, given a sequence of o/p data from a system, one is interested in modeling the signal as the o/p of a linear time invariant system and analyzing the spectral content.

'Spectral content' is linked to "system mode".

Example: Suppose we have a difference eqn given by

$$y(t+2) + a_1 y(t+1) + a_2 y(t) = 0 \quad \text{--- (1)}$$

This is a homogenous 2^{nd} order system

Taking Z-transforms on b. s. of (1), we get a characteristic eqn

$$\left(\therefore \left(z^2 + a_1 z + a_2\right) Y(z) = 0\right)$$

$$\left(z^2 + a_1 z + a_2\right) = 0$$

$$z = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_2}}{2}$$

$$p_1 = \frac{-a_1 + \sqrt{a_1^2 - 4a_2}}{2} \qquad p_2 = \frac{-a_1 - \sqrt{a_1^2 - 4a_2}}{2}$$

$$z^2 + a_1 z + a_2 \equiv (z - p_1)(z - p_2)$$

<u>Case A:</u> $p_1 \neq p_2$

$$y(t) = c_1 (p_1)^t + c_2 (p_2)^t \qquad t \geq 0$$

c_1 & c_2 can be determined from the initial Conditions

<u>CASE B:</u> $p_1 = p_2 = p$

$$y(t) = (c_1 + c_2 t)\, p^t$$

Example : Suppose we have a mixture of 2 sinusoids & observations are "noise free".

$$y(t) = a_1 \cos(\omega_1 t) + a_2 \cos(\omega_2 t)$$

We need to determine the mode frequencies.

$$\cos(\omega_i t) = \frac{e^{j\omega_i t} + e^{-j\omega_i t}}{2}$$

Each $\cos(\omega_i t)$ has 2 modes!

$\Rightarrow$ $y(t)$ is governed by a 4th order difference eqn.

Let us set up the recursive eqn

$$y(t) + \sum_{i=1}^{4} c_i \, y(t-i) = 0$$

$$\underbrace{\begin{bmatrix} -y(3) & -y(2) & -y(1) & -y(0) \\ & \ddots & & \\ -y(6) & -y(5) & -y(4) & -y(3) \end{bmatrix}} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} y(4) \\ y(5) \\ y(6) \\ y(7) \end{bmatrix}$$

$$\left(\because \; y(4) + c_1 y(3) + c_2 y(2) + c_3 y(1) + c_4 y(0) = 0 \right.$$
$$\vdots$$

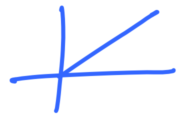
Can solve for $[c_1 \;\; c_2 \;\; c_3 \;\; c_4]^T$

We need 8 measurements for 4 modes !

Home Work

1) What are the modes for the linear ramp (noise free)?

$\{0, 1, 2, \ldots\}$



2) Repeat Problem 1 for the signal

$\left\{1, \frac{3}{4}, \frac{1}{2}, \frac{5}{16}, \ldots\right\}$

3) Develop a simple Matlab model for the generalized mixture of sinusoids. Experiment with your choice of frequencies. Can you determine the amplitudes & the phases from the initial conditions?

$$y(t) = \sum_{i=1}^{n} a_i \cos(w_i t + \theta_i)$$

Linear discrete time models

The general equation for a linear discrete time model is (ARMA) given by

$$\sum_{k=0}^{p} a_k\, y(n-k) = \sum_{k=0}^{q} b_k\, f(n-k) \qquad (1)$$

y: o/p

f: forcing function or i/p

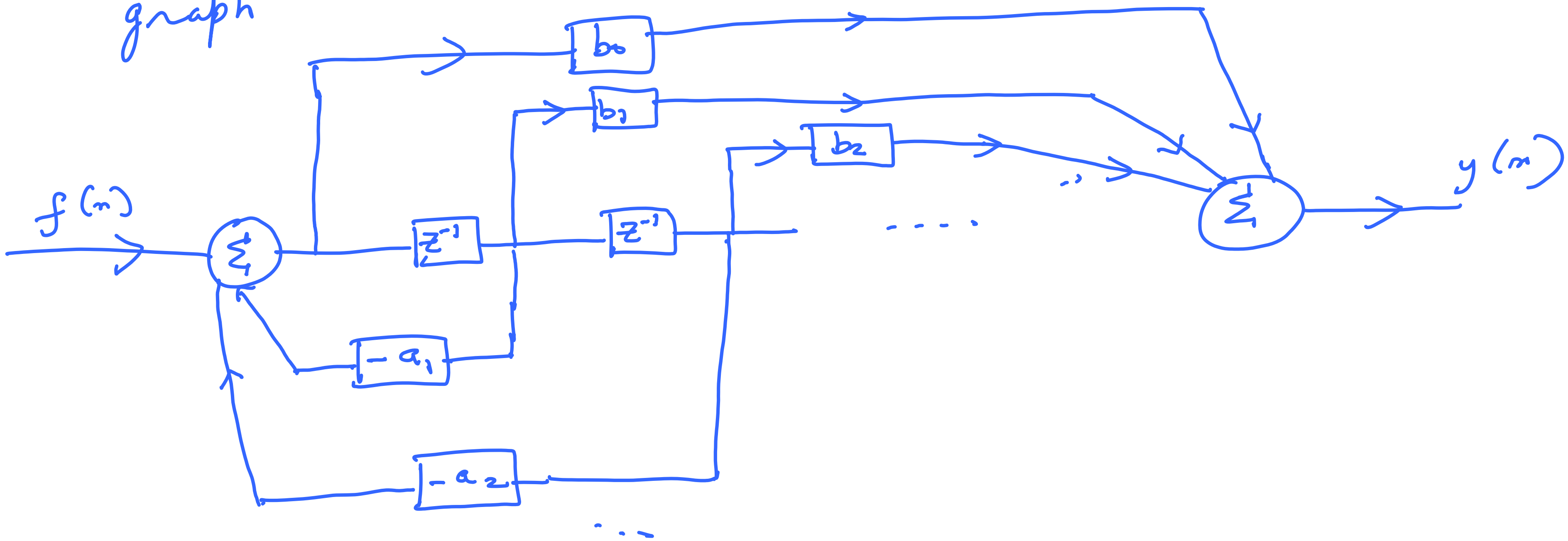
When $p = 0$, eqn (1) is a moving average signal since we scale the i/p over a $(q+1)$ window

When $q = 0$, with $a_0 = 1$,

$$y(n) = b_0 f(n) - \sum_{k=1}^{p} a_k\, y(n-k)$$

(AUTO REGRESSIVE MODEL of order 'p')

Let us try to realize the system as a signal flow graph



The above set up is useful for state space representation

Generalized state space model
Linear discrete time models

Consider the linear discrete time model with transfer function for (p=q) case.

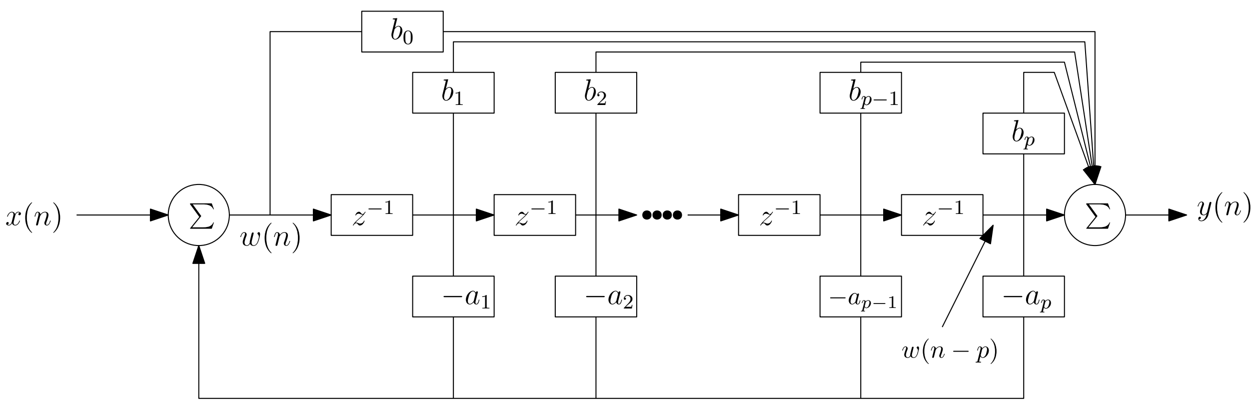
$$H(z) = \frac{\sum_{k=0}^{p} b_k z^{-k}}{1 + \sum_{k=1}^{P} a_k z^{-k}} = \frac{Y(z)}{X(z)}$$

Let us define two related transfer functions as follows

$$\frac{Y(z)}{W(z)} = \sum_{k=0}^{p} b_k z^{-k}$$

$$\frac{W(z)}{X(z)} = \frac{1}{1 + \sum_{k=1}^{p} a_k z^{-k}}$$

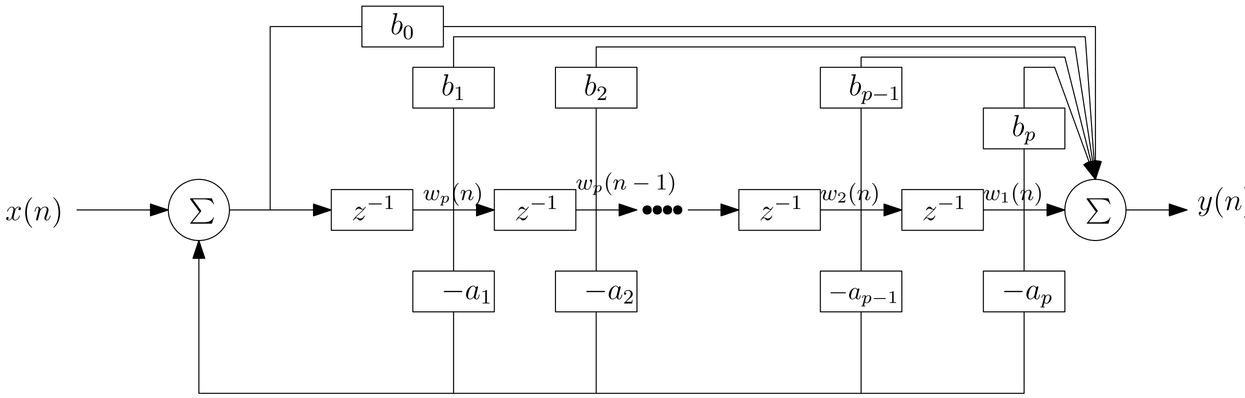
Let us form the signal flow graph for representing transfer functions above.



Define the state variables as follows:

$$\begin{aligned} w_p(n) &= w(n-1) \\ w_{p-1}(n) &= w(n-2) \\ &\vdots \\ w_1(n) &= w(n-p) \end{aligned}$$

As the signal $w(n)$ passes through the delay line, the state variables $[w_1(n), \ldots, w_p(n)]$ form a vector. The time to space mapping dictates that the signal in time can be transformed to a vector in space. The signal



dynamics can be visualized as a **trajectory** as below.

$$
\begin{aligned}
w_0(n+1) &= w_1(n)\\
w_1(n+1) &= w_2(n)\\
&\vdots\\
w_{p-1}(n+1) &= w_p(n)\\
w_p(n+1) &= x(n) - a_1 w_p(n) - a_2 w_{p-1}(n) - \cdots - a_p w_1(n)
\end{aligned}
$$

Let us form a state vector $\underline{W}(n) = [w_1(n), \ldots, w_p(n)]^{\mathrm{T}}$. Using this and the above expressions, we have

$$\underline{W}(n+1) = \mathbf{A}\underline{W}(n) + \mathbf{b}x(n)$$

where

$$
\mathbf{A} = \begin{bmatrix}
0 & 1 & 0 & 0 & \ldots & 0 & 0\\
0 & 0 & 1 & 0 & \ldots & 0 & 0\\
\vdots & & & & & & \\
-a_p & -a_{p-1} & -a_{p-2} & -a_{p-3} & \ldots & -a_2 & -a_1
\end{bmatrix},
$$

$$\mathbf{b} = [\underbrace{0, 0, \ldots, 0, 1}_{p \text{ elements}}]^{\mathrm{T}}$$

Similarly, one can do the math for expressing the output $y(n)$ through a sequence of equations below:

$$
\begin{aligned}
y(n) &= b_0 w(n) + \sum_{k=1}^{p} b_k w_{p+1-k}(n)\\
y(n) &= b_0 w_p(n+1) + \sum_{k=1}^{p} b_k w_{p+1-k}(n)\\
y(n) &= b_0[x(n) - a_1 w_p(n) - a_2 w_{p-1}(n) - \cdots - a_p w_1(n)] + b_1 w_p(n) + b_2 w_{p-1}(n) + \cdots + b_p w_1(n)\\
y(n) &= \sum_{k=1}^{p} [b_k - b_0 a_k] w + b_0 x(n)\\
y(n) &= \mathbf{c}^{\mathrm{T}}\underline{W}(n) + \mathbf{d}x(n)
\end{aligned}
$$

where

$$\mathbf{c} = \begin{bmatrix} b_p - b_0 a_p \\ \vdots \\ b_1 - b_0 a_1 \end{bmatrix},$$

$$\mathbf{d} = b_0.$$

In the time domain,

$$\underline{w}(n) = A^n \underline{w}(0) + \sum_{k=0}^{n-1} A^k \underline{b}\, x(n-1-k)$$

The o/p is

$$y(n) = \underline{c}^T \underline{w}(n) + d\, x(n)$$

Exercise: Suppose $H(z) = \dfrac{1 + 2z^{-1} + z^{-2}}{1 - 0.75 z^{-1} + 0.125 z^{-2}}$

and $x[n] = \left(\frac{1}{2}\right)^n u(n)$

1) Obtain the state variable representation & study the o/p response
(a) mathematically (b) via simulations
2) Verify these results by any of your favourite undergrad. methods studied in DSP

Derivation of the transfer function from state variable representation

$$\underline{w}(n+1) = A\,\underline{w}(n) + \underline{b}\,x(n) \quad \text{--- (a)}$$
$$y(n) = \underline{c}^T\,\underline{w}(n) + d\,x(n) \quad \text{--- (b)}$$

$$\frac{Y(z)}{X(z)} = H(z) = \underline{c}^T (zI - A)^{-1}\,\underline{b} + d$$

$$z\,\underline{W}(z) = A\,\underline{W}(z) + \underline{b}\,X(z) \quad \text{--- } (1)$$
$$Y(z) = \underline{c}^T\,W(z) + d\,X(z) \quad \text{--- } (2)$$

Rewriting (1),

$$(zI - A)\,\underline{W}(z) = \underline{b}\,X(z) \quad \text{--- } (3)$$

Plug (3) in (2)

$$Y(z) = \left(\underline{c}^T (zI - A)^{-1}\,\underline{b} + d\right) X(z) \quad \text{--- } (4)$$

Non-unique state representations

For any invertible $p \times p$ matrix T, we can form a different state representation

State variable representation is 'not' unique.

$$\underline{W}(n) = T\,\underline{z}(n) \; ; \qquad W = Tz$$

$$T\,\underline{z}(n+1) = A\,T\,\underline{z}(n) + \underline{b}\,x(n) \quad \text{———} \quad (i)$$

$$y(n) = \underline{c}^T\,T\,\underline{z}(n) + d\,x(n) \quad \text{———} \quad (ii)$$

Rewriting (i) in a slightly different way,

$$\underline{z}(n+1) = T^{-1}A\,T\,\underline{z}(n) + T^{-1}\underline{b}\,x(n) \quad \text{———} \quad (iii)$$

$$y(n) = \underline{c}^T\,T\,\underline{z}(n) + d\,x(n) \quad \text{———} \quad (iv)$$

$$(\tilde{A}, \underline{\tilde{b}}, \underline{\tilde{c}}, \tilde{d}) = (T^{-1}AT,\; T^{-1}\underline{b},\; T^T\underline{c},\; d)$$

$(a, b, c) \longrightarrow (b, c, a)$

$$\begin{bmatrix} b \\ c \\ a \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

If we 'permute' the state variables, we get a non unique representation "trivial"

What characterizes the "Similarity" between various state representations?

<u>Eigen values</u>

1. State variable formulation & defn. of a state vector
2. Derive the transfer function.
3. State space model (Analyze the dynamics for various forcing functions (i/ps))
4. State space models need not be unique

Vector Spaces

A finite dimensional vector may be written as
$\underset{\to}{x} = [x_1 \ x_2 \ \cdots \ x_n]^T$. The elements are $x_1 \ x_2 \ \cdots \ x_n$
Each of the elements $\in$ some set such as $\mathbb{R}$ i.e., $x_i \in \mathbb{R}$
or $x_i \in \mathbb{F}_2$. They are the scalars of the vector space.

__Definition__ : A linear vector space S over a set of scalars
R is a collection of objects known as "__vectors__" together with
an additive (+) operation and scalar multiplication ($\cdot$)
satisfying the following properties :

VS 1: S forms a group under addition

(a) For any $\underline{x}$ and $\underline{y} \in S$, $\underline{x} + \underline{y} \in S$

(b) There is an <u>identity element</u> in S denoted by $\underline{0}$ /

$\underline{x} + \underline{0} = \underline{0} + \underline{x} = \underline{x}$

(c) For every element $\underline{x} \in S$, there is another element

$\underline{y} \in S$ / $\underline{x} + \underline{y} = \underline{0}$

$\underline{y} = -\underline{x}$ is the "<u>additive</u>" inverse

(d) Addition operation is associative. For any $\underline{x}, \underline{y}, \underline{z} \in S$

$(\underline{x} + \underline{y}) + \underline{z} = \underline{x} + (\underline{y} + \underline{z})$

VS2: For any a and $b \in R$, $\underline{x}, \underline{y} \in S$

$a\underline{x} \in S$

$a(b\underline{x}) = (ab)\underline{x}$

$(a+b)\underline{x} = a\underline{x} + b\underline{x}$

$a(\underline{x}+\underline{y}) = a\underline{x} + a\underline{y}$

VS3: There is a multiplicative element (identity) $1 \in \mathbb{R}$ such that $1 \cdot \underline{x} = \underline{x}$, $0 \in R \,/\, 0 \cdot \underline{x} = \underline{0}$.

Examples : A most familiar vector space is R^n; set of all n tuples

$$\underline{x}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \in R^3 \quad \underline{x}_2 = \begin{bmatrix} 0 \\ 1 \\ 8 \end{bmatrix} \in R^3 \quad \ldots$$

Other examples :

1) Set of $m \times n$ matrices with real entries.

2) Set of all polynomials up to degree n with real coeffts.

Infinite dimensional vector spaces

Examples:

1) Sequence spaces: Set of all ∞-long sequences $\{x_n\}$

2) Set of continuous functions defined over the interval $[a, b]$ etc.

Defn: Let S be a vector space. If $V \subset S$ is a subset / V itself is a vector space, then V is called a subspace of S.

Examples: (From codes)

Let $S = \{(0\,0\,0\,0\,0), (0\,1\,0\,0\,1), (1\,0\,0\,0\,1), (1\,1\,0\,0\,0)\}$

$+$ operation is modulo 2.

$V = \{(0\,0\,0\,0\,0), (0\,1\,0\,0\,1)\}$ Is V a subspace of S?

'Signals' can be thought of vectors in a signal space

The notion of V.S. can be naturally extended to signals.