Homework #5 solution key

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Problem 8.5.

Solution. For the matched filter considered in Example 2, it is given that the random vector $\mathbf{X} = \mathbf{s} + \mathbf{V}$, where the signal component is represented by $\mathbf{s}$ has a fixed Euclidean norm of one, i.e., $\sqrt{\mathbf{s}^{\mathrm{T}}\mathbf{s}} = 1 \implies \mathbf{s}^{\mathrm{T}}\mathbf{s} = 1$.
The random vector $\mathbf{V}$, represents the additive noise component and has zero mean and covariance matrix $\sigma^2\mathbf{I}$. The correlation matrix of $\mathbf{X}$ is given by $\mathbf{R} = \mathbf{s}\mathbf{s}^{\mathrm{T}} + \sigma^2\mathbf{I}$.
The largest eigenvalue of the correlation matrix $\mathbf{R}$ and the corresponding eigenvector are $\lambda_1 = 1 + \sigma^2$ and $\mathbf{q}_1 = \mathbf{s}$ respectively. Let us post multiply the correlation matrix by the eigenvector $\mathbf{q}_1$.

$$\begin{aligned}
\mathbf{R}\mathbf{q}_1 &= \left(\mathbf{s}\mathbf{s}^{\mathrm{T}} + \sigma^2\mathbf{I}\right)\mathbf{q}_1 \\
&= \left(\mathbf{s}\mathbf{s}^{\mathrm{T}} + \sigma^2\mathbf{I}\right)\mathbf{s} \\
&= \mathbf{s}\mathbf{s}^{\mathrm{T}}\mathbf{s} + \sigma^2\mathbf{s} \\
&= \mathbf{s} + \sigma^2\mathbf{s} \qquad (\text{since } \mathbf{s}^{\mathrm{T}}\mathbf{s} = 1) \\
&= \left(1 + \sigma^2\right)\mathbf{s} \\
&= \lambda_1\mathbf{s} \\
&= \lambda_1\mathbf{q}_1
\end{aligned}$$

Therefore the given parameters satisfy the basic relation of $\mathbf{R}\mathbf{q}_1 = \lambda_1\mathbf{q}_1$. ■

Problem 8.15.

Solution. Let the total number of inputs be N. The data $\mathbf{x}_i$ is projected using a kernel to get the projected data set $\phi(\mathbf{x}_i)$. Let $\tilde{\phi}(\mathbf{x}_i)$ be the projected data points after centralizing the data as given below

$$\tilde{\phi}(\mathbf{x}_i) = \phi(\mathbf{x}_i) - \frac{1}{N}\sum_{l=1}^{N}\phi(\mathbf{x}_l).$$

We can get the corresponding elements of the gram matrix $\tilde{K}$ as below:

$$\begin{aligned}
\tilde{k}_{ij} &= \tilde{\phi}^{\mathrm{T}}(\mathbf{x}_i)\tilde{\phi}(\mathbf{x}_j) \\
&= \left(\phi(\mathbf{x}_i) - \frac{1}{N}\sum_{m=1}^{N}\phi(\mathbf{x}_m)\right)^{\mathrm{T}}\left(\phi(\mathbf{x}_j) - \frac{1}{N}\sum_{n=1}^{N}\phi(\mathbf{x}_n)\right) \\
&= \phi(\mathbf{x}_i)^{\mathrm{T}}\phi(\mathbf{x}_j) - \frac{1}{N}\sum_{m=1}^{N}\phi^{\mathrm{T}}(\mathbf{x}_m)\phi(\mathbf{x}_j) - \frac{1}{N}\sum_{n=1}^{N}\phi^{\mathrm{T}}(\mathbf{x}_i)\phi(\mathbf{x}_n) + -\frac{1}{N^2}\sum_{m=1}^{N}\sum_{n=1}^{N}\phi^{\mathrm{T}}(\mathbf{x}_m)\phi(\mathbf{x}_n) \\
&= k_{ij} - \frac{1}{N}\sum_{m=1}^{N}k_{mj} - \frac{1}{N}\sum_{n=1}^{N}k_{in} + -\frac{1}{N^2}\sum_{m=1}^{N}\sum_{n=1}^{N}k_{mn}
\end{aligned}$$

A compact representation of the above in the matrix form is given by

$$\tilde{\mathbf{K}} = \mathbf{K} - \mathbf{NK} - \mathbf{KN} + \mathbf{NKN}$$

where $\mathbf{N}$ is an $N \times N$ matrix with all entries as $\frac{1}{N}$. ■

Problem 8.16.

Solution. It is given that the eigenvector $\tilde{q}$ of the correlation matrix $\tilde{\mathbf{R}}$ is normalized to unit length, that is,

$$\tilde{q}_k^{\mathrm{T}}\tilde{q}_k = 1 \quad \text{for } k = 1, 2, \ldots, l \tag{1}$$

where it is assumed that the eigenvalues of $\mathbf{K}$ are arranged in descending order with λ_l being the smallest nonzero eigenvalue of the Gram matrix $\mathbf{K}$. We know that

$$\tilde{q} = \sum_{j=1}^{N}\alpha_j\phi(x_j) \tag{2}$$

$$K\alpha = \lambda\alpha \tag{3}$$

Using (2) in (1) and upon simplifying, we get

$$\begin{aligned}
1 &= \tilde{q}_k^{\mathrm{T}}\tilde{q}_k \\
&= \sum_{i=1}^{N}\sum_{j=1}^{N}\alpha_{ki}^{\mathrm{T}}\alpha_{kj}\phi(x_i)^{\mathrm{T}}\phi(x_j) \\
&= \alpha_k^{\mathrm{T}}K\alpha_k \\
&= \alpha_k^{\mathrm{T}}\lambda_k\alpha_k \quad \text{(Using (3))} \\
&= \lambda_k\alpha_k^{\mathrm{T}}\alpha_k \\
\frac{1}{\lambda_k} &= \alpha_k^{\mathrm{T}}\alpha_k \quad \text{for } k = 1, 2, \ldots, l.
\end{aligned}$$

Therefore, we see that normalization of the eigenvector α of the Gram matrix $\mathbf{K}$ is equivalent to the requirement of Eq(8.109) to be satisfied.

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