

**PROBLEM 1:**

(a) Let  $H(z) = \frac{1+az^{-1}}{a+z^{-1}}$ ,  $a \in \mathbb{R}$ . Write down the expressions for the Type 1 polyphase components (with  $M = 2$ ). What can you say about  $H(z)$  for various values of  $a$ ? (4 points)

(b) Let  $H(z) = \frac{1}{1-2R\sin\theta z^{-1}+R^2 z^{-2}}$  with  $R > 0$  and  $\theta \in \mathbb{R}$ . Find the Type 1 polyphase components for  $M = 2$ . (4 points)

**Solution:**

Writing the polyphase decomposition of  $H(z)$ :

$$H(z) = E_0(z^2) + z^{-1}E_1(z^2)$$

(a)

$$\begin{aligned} H(z) &= \frac{1+az^{-1}}{a+z^{-1}}, \quad a \in \mathbb{R} \\ &= \frac{(1+az^{-1})(a-z^{-1})}{a^2-z^{-2}} = \frac{a(1-z^{-2})}{a^2-z^{-2}} + z^{-1} \frac{a^2-1}{a^2-z^{-2}}. \\ \therefore E_0(z) &= \frac{a(1-z^{-1})}{a^2-z^{-1}} \text{ and } E_1(z) = \frac{a^2-1}{a^2-z^{-1}}. \end{aligned}$$

This gives the type-I decomposition of  $H(z)$ .

$$|H(e^{i\omega})| = 1 \Rightarrow \text{It is an all pass filter } \forall a.$$

For  $a = 0$ ,  $H(z) = z \Rightarrow$  It is a non causal filter. For  $a > 1$ ,  $H(z)$  is stable and for  $a < 1$ ,  $H(z)$  is unstable.

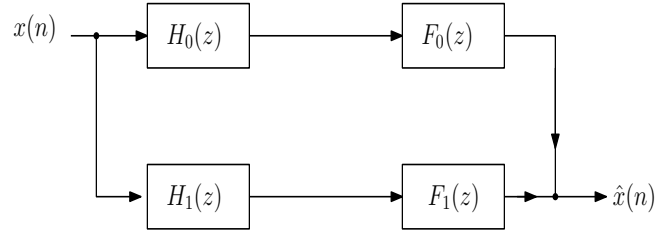
(b)

$$\begin{aligned} H(z) &= \frac{1}{1-2R\sin\theta z^{-1}+R^2 z^{-2}}, \quad R > 0, \theta \in \mathbb{R} \\ &= \frac{(1+R^2 z^{-2})+2R\sin\theta z^{-1}}{(1+R^2 z^{-2})^2-4R^2\sin^2\theta z^{-2}} = \frac{1+R^2 z^{-2}}{1+2R^2\cos 2\theta z^{-2}+R^4 z^{-4}} + z^{-1} \frac{2R\sin\theta}{1+2R^2\cos 2\theta z^{-2}+R^4 z^{-4}}. \\ \therefore E_0(z) &= \frac{1+R^2 z^{-1}}{1+2R^2\cos 2\theta z^{-1}+R^4 z^{-2}} \text{ and } E_1(z) = \frac{2R\sin\theta}{1+2R^2\cos 2\theta z^{-1}+R^4 z^{-2}}. \end{aligned}$$

This gives the type-I decomposition of  $H(z)$ .

**PROBLEM 2:**

Consider the analysis/synthesis system shown below:



(a) Let the analysis filters be  $H_0(z) = 1 + 3z^{-1} + \frac{1}{3}z^{-2} + z^{-3}$  and  $H_1(z) = H_0(-z)$ . Find causal stable IIR filters  $F_0(z)$  and  $F_1(z)$  such that  $\hat{x}(n)$  agrees with  $x(n)$  except for a possible delay and (non zero) scale factor. (5 points)

(b) Let the analysis filters be  $H_0(z) = 1 + z^{-1} + 3z^{-2} + z^{-3} + z^{-4}$  and  $H_1(z) = H_0(-z)$ . Find causal stable FIR filters  $F_0(z)$  and  $F_1(z)$  such that  $\hat{x}(n)$  agrees with  $x(n)$  except for a possible delay and (non zero) scale factor. (5 points)

**Solution:**

From the block diagram, we observe that

$$\hat{X}(z) = (H_0(z)F_0(z) + H_1(z)F_1(z))X(z)$$

Writing the polyphase decomposition of  $H_0(z)$ :

$$\begin{aligned} H_0(z) &= E_0(z^2) + z^{-1}E_1(z^2) \\ H_1(z) &= H_0(-z) = E_0(z^2) - z^{-1}E_1(z^2) \\ \Rightarrow \frac{\hat{X}(z)}{X(z)} &= E_0(z^2)(F_0(z) + F_1(z)) + z^{-1}E_1(z^2)(F_0(z) - F_1(z)) \end{aligned}$$

(a)

$$\begin{aligned} H_0(z) &= 1 + 3z^{-1} + \frac{1}{3}z^{-2} + z^{-3} \\ E_0(z^2) &= 1 + \frac{1}{3}z^{-2} \\ z^{-1}E_1(z^2) &= 3z^{-1} + z^{-3} \\ \Rightarrow \frac{\hat{X}(z)}{X(z)} &= (1 + \frac{1}{3}z^{-2})(F_0(z) + F_1(z)) + (3z^{-1} + z^{-3})(F_0(z) - F_1(z)) \end{aligned}$$

We can choose  $F_0(z) = F_1(z) = \frac{1}{2(1+\frac{1}{3}z^{-2})} \Rightarrow \frac{\hat{X}(z)}{X(z)} = 1$ . Note that  $F_0(z)$  and  $F_1(z)$  are causal and stable because the poles are located at  $\pm j\frac{1}{\sqrt{3}}$  (inside the unit circle).

(b)

$$\begin{aligned} H_0(z) &= 1 + z^{-1} + 3z^{-2} + z^{-3} + z^{-4} \\ E_0(z^2) &= 1 + 3z^{-2} + z^{-4} \\ z^{-1}E_1(z^2) &= z^{-1} + z^{-3} = z^{-1}(1 + z^{-2}) \\ \Rightarrow \frac{\hat{X}(z)}{X(z)} &= (1 + 3z^{-2} + z^{-4})(F_0(z) + F_1(z)) + z^{-1}(1 + z^{-2})(F_0(z) - F_1(z)) \\ &= ((1 + z^{-2})^2 + z^{-2})(F_0(z) + F_1(z)) + z^{-1}(1 + z^{-2})(F_0(z) - F_1(z)) \\ &= (1 + z^{-2})^2(F_0(z) + F_1(z)) + z^{-2}(F_0(z) + F_1(z)) + z^{-1}(1 + z^{-2})(F_0(z) - F_1(z)) \end{aligned}$$

We can choose  $F_0(z) + F_1(z) = z^{-1}$  and  $F_0(z) - F_1(z) = -(1 + z^{-2})$ .

$$\Rightarrow \frac{\hat{X}(z)}{X(z)} = z^{-1}(1 + z^{-2})^2 + z^{-3} - z^{-1}(1 + z^{-2})^2 = z^{-3}.$$

Therefore, perfect reconstruction is possible with the following causal FIR filters:

$$F_0(z) = \frac{1}{2}(-1 + z^{-1} - z^{-2});$$

$$F_1(z) = \frac{1}{2}(1 + z^{-1} + z^{-2}).$$

**PROBLEM 3:**

Let  $H_0(z) = \frac{1+2z^{-1}}{2}$ . Find the real coefficient causal FIR filter  $H_1(z)$  such that the pair  $(H_0(z), H_1(z))$  is power complementary. Are these filters also all pass complementary? (6 points)

**Solution:**

Since  $H_0(z)$  and  $H_1(z)$  are power complementary, we have:

$$\begin{aligned} |H_0(e^{j\omega})|^2 + |H_1(e^{j\omega})|^2 &= K \text{ (some constant)} \\ |1/2 + e^{-j\omega}|^2 + |H_1(e^{j\omega})|^2 &= K \\ (1/2 + \cos\omega)^2 + \sin^2\omega + |H_1(e^{j\omega})|^2 &= K \\ \frac{1}{4} + 1 + \cos\omega + |H_1(e^{j\omega})|^2 &= K \end{aligned}$$

By choosing  $H_1(z) = \frac{1}{2} - z^{-1}$ , we get

$$|H_0(e^{j\omega})|^2 + |H_1(e^{j\omega})|^2 = \frac{5}{4} + \frac{5}{4} = \frac{5}{2}.$$

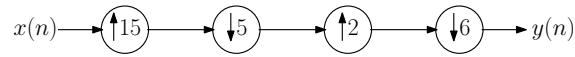
We check for all pass complementarity:

$$H_0(z) + H_1(z) = \frac{1}{2} + z^{-1} + \frac{1}{2} - z^{-1} = 1.$$

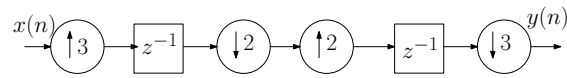
Therefore,  $(H_0(z), H_1(z))$  are all pass complementary.

**PROBLEM 4:**

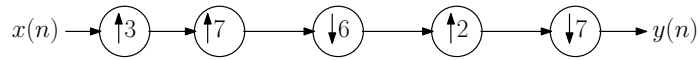
Simplify the following multirate systems shown below as best as you can. Obtain the z-transform of the output signal in terms of that of the input signal. ( $3 \times 4 = 12$  points)



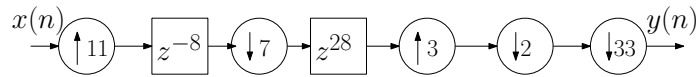
(a)



(b)



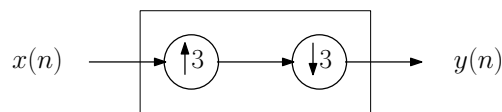
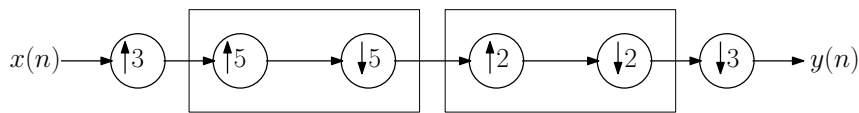
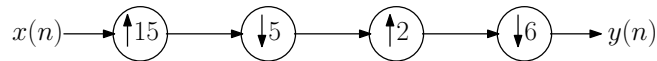
(c)



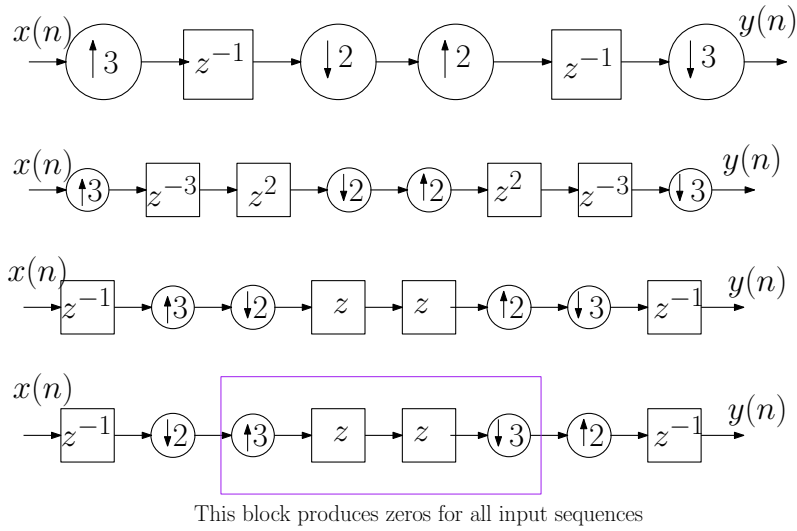
(d)

**Solution:**

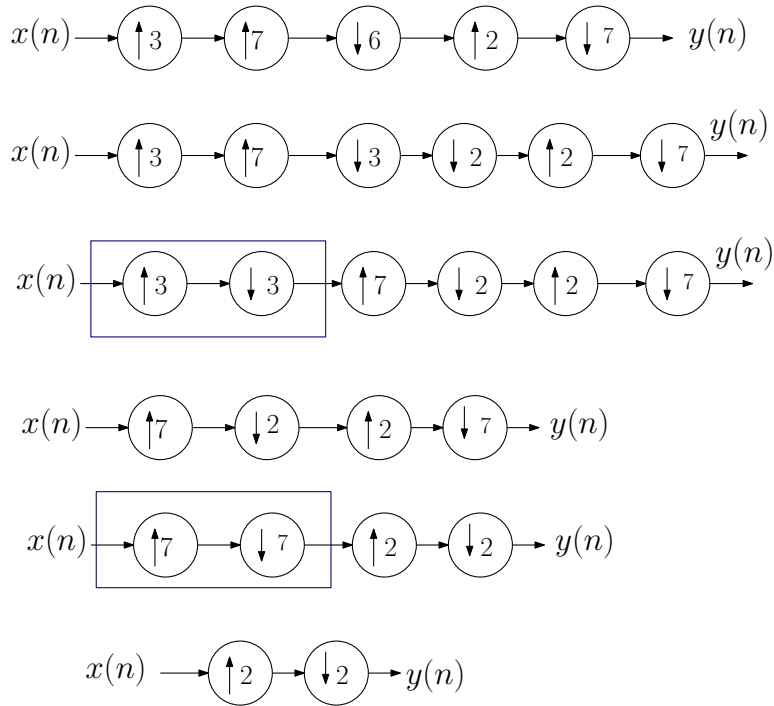
(a)  $y(n) = x(n)$ .



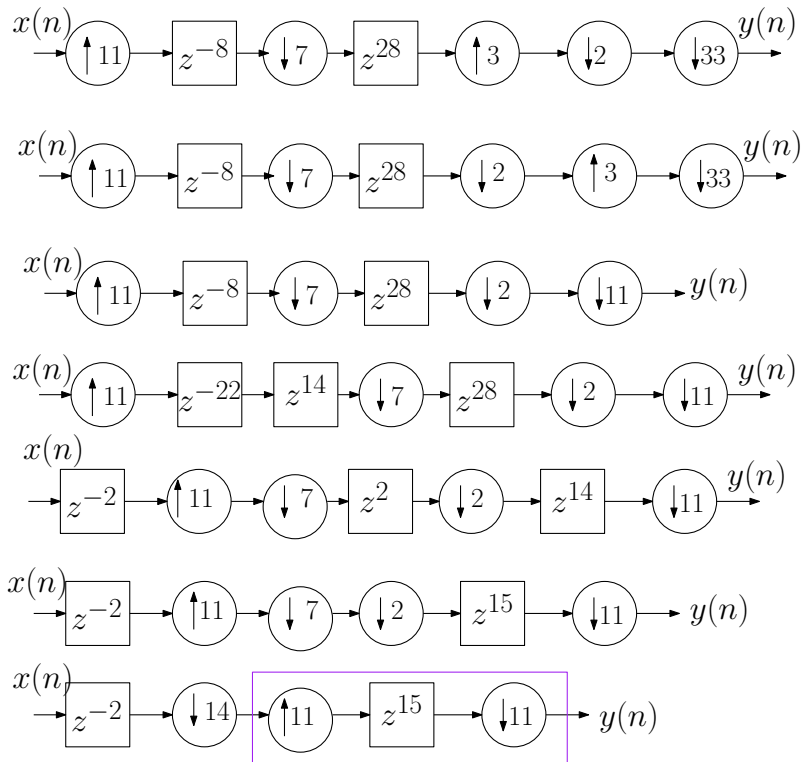
(b)  $y(n) = 0$



(c)  $Y(z) = \frac{1}{2}(X(z) + X(-z))$ .



(d)  $y(n) = 0$



This block produces all zeros for all inputs

**PROBLEM 5:**

Prove that decimation by  $M$  followed by expansion by  $L$  can be interchanged if  $L$  and  $M$  are relatively prime. You must prove this result in the time and frequency domain representations. (10 points)

**Solution:**

**(Frequency domain analysis)** From Figure 2,

$$\begin{aligned} X_1(z) &= \frac{1}{M} \sum_{i=0}^{M-1} X\left(z^{\frac{1}{M}} e^{j\frac{2\pi i}{M}}\right) \\ \implies Y_1(z) &= X_1(z^L) \\ &= \frac{1}{M} \sum_{i=0}^{M-1} X\left(z^{\frac{L}{M}} e^{j\frac{2\pi i}{M}}\right). \end{aligned} \quad (1)$$

Similarly,

$$\begin{aligned} X_2(z) &= X(z^L) \\ Y_2(z) &= \frac{1}{M} \sum_{i=0}^{M-1} X_2\left(z^{\frac{1}{M}} e^{j\frac{2\pi i}{M}}\right) \\ &= \frac{1}{M} \sum_{i=0}^{M-1} X\left(\left(z^{\frac{1}{M}} e^{j\frac{2\pi i}{M}}\right)^L\right) \\ &= \frac{1}{M} \sum_{i=0}^{M-1} X\left(z^{\frac{L}{M}} e^{j\frac{2\pi iL}{M}}\right). \end{aligned} \quad (2)$$

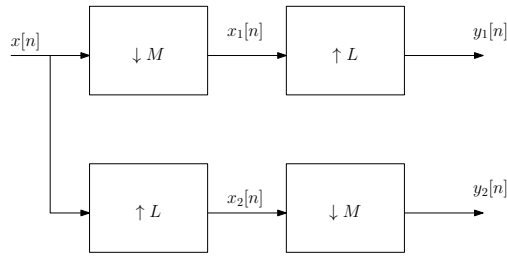


Figure 2: Comparing the outputs by changing the order of decimator and upsampler.

To prove that  $Y_1(z) = Y_2(z) \forall X(z)$ , it is necessary and sufficient to satisfy the following condition:

$$\begin{aligned} \left\{ X\left(z^{\frac{L}{M}} e^{j\frac{2\pi iL}{M}}\right) \mid i = 0, 1, \dots, M-1 \right\} &= \left\{ X\left(z^{\frac{1}{M}} e^{j\frac{2\pi i}{M}}\right) \mid i = 0, 1, \dots, M-1 \right\} \forall X(z) \\ \text{i.e., } \left\{ e^{j\frac{2\pi iL}{M}} \mid i = 0, 1, \dots, M-1 \right\} &= \left\{ e^{j\frac{2\pi i}{M}} \mid i = 0, 1, \dots, M-1 \right\}. \end{aligned}$$

Since  $e^{j2\pi k} = 1 \forall k \in \mathbb{Z}$ , we have  $e^{j\frac{2\pi iL}{M}} = e^{j\frac{2\pi(iL \bmod M)}{M}}$ . Hence, the equivalent condition is

$$\{(iL) \bmod M \mid i = 0, 1, \dots, M-1\} = \{0, 1, \dots, M-1\}. \quad (3)$$

Let  $0 \leq i_1 \leq M-1$  and  $0 \leq i_2 \leq M-1$  such that  $i_1 \neq i_2$ . Without loss of generality, consider  $i_1 < i_2$ . Using the following identity on modulo operation

$$(a - b) \bmod M = (a \bmod M - b \bmod M) \bmod M,$$



we have,

$$((i_1L) \bmod M - (i_2L) \bmod M) \bmod M = ((i_1 - i_2)L) \bmod M. \quad (4)$$

**Case  $L$  and  $M$  are relatively prime:**

Since  $0 < i_1 - i_2 < M$ , and  $\gcd(L, M) = 1$ ,  $((i_1 - i_2)L) \bmod M \neq 0$ . Therefore from (4),

$$\begin{aligned} ((i_1L) \bmod M - (i_2L) \bmod M) \bmod M &\neq 0, \\ \implies (i_1L) \bmod M &\neq (i_2L) \bmod M. \end{aligned}$$

We have proved that  $i_1 \neq i_2 \implies (i_1L) \bmod M \neq (i_2L) \bmod M \forall i_1, i_2 \in \{0, 1, 2, \dots, M-1\}$ . Therefore, when  $\gcd(L, M) = 1$ , equation (3) holds true.

**Case  $M$  divides  $L$ :**

Let  $L = P \times M$ ,  $P > 1$ . Therefore, it is possible to chose  $i_1 = i_2 + M$ . Under this condition,

$$((i_1 - i_2)L) \bmod M = (ML) \bmod M = 0.$$

Therefore,

$$\begin{aligned} ((i_1L) \bmod M - (i_2L) \bmod M) \bmod M &= 0 \\ \implies (i_1L) \bmod M &= (i_2L) \bmod M. \end{aligned}$$

We have shown that for some choice of  $i_1 \neq i_2$ ,  $(i_1L) \bmod M = (i_2L) \bmod M$ . Hence, the values  $\{(iL) \bmod M\}_{i=0}^{M-1}$  are not distinct. Therefore, when  $M$  divides  $L$ , equation (3) does not hold true.

**Case  $\gcd(M, L) = G > 1$ :**

Let  $M = G \times P_M$  and  $L = G \times P_L$ . We can chose  $i_1 = i_2 + G$ . Under this condition,  $e^{j2\pi \frac{i_1L}{M}} = e^{j2\pi \frac{i_2L}{P_M}}$ . Therefore,  $\left\{ e^{j2\pi \frac{iP_L}{P_M}} \mid i = 0, 1, \dots, M-1 \right\}$  has  $P_M$  distinct values. Therefore, equation (3) does not hold true under this condition.

Hence, the equation (3) holds true iff  $L$  and  $M$  are relatively prime. This proves that  $M$  fold decimator and  $L$  fold upsampler blocks can be interchanged iff  $L$  and  $M$  are relatively prime.

**(Time domain analysis)** From the definitions of decimator and upsampler,

$$\begin{aligned} x_1[n] &= x[Mn]. \\ y_1[n] &= \begin{cases} x_1\left[\frac{n}{L}\right], & n \text{ is a multiple of } L \\ 0 & \text{otherwise,} \end{cases} \\ y_1[n] &= \begin{cases} x\left[M\frac{n}{L}\right], & n \text{ is a multiple of } L \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \quad (5)$$

Similarly,

$$\begin{aligned} x_2[n] &= \begin{cases} x_1\left[\frac{n}{L}\right], & n \text{ is a multiple of } L \\ 0 & \text{otherwise.} \end{cases} \\ y_1[n] &= x_1[Mn], \\ y_1[n] &= \begin{cases} x\left[\frac{Mn}{L}\right], & Mn \text{ is a multiple of } L \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \quad (6)$$

From equations (5) and (6), the outputs are same iff  $n$  is a multiple of  $L$  when ever  $Mn$  is a multiple of  $L$ .

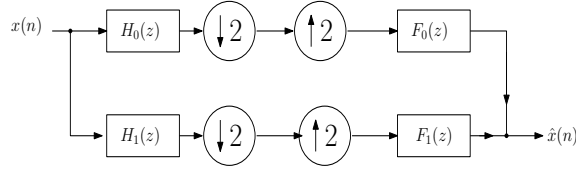
**Case  $\gcd(L, M) = 1$ :** Trivial in this case that  $L$  divides  $Mn \iff L$  divides  $n$ .

**Case  $\gcd(L, M) = P \neq 1$ :** Let  $L = P \times Q$ . In this case  $L$  divides  $Mn$  when ever  $Q$  divides  $n$ . Hence  $L$  divides  $Mn \not\Rightarrow L$  divides  $n$ .

Therefore, the outputs are same iff  $L$  and  $M$  are relatively prime.

**PROBLEM 6:**

Consider the two channel QMF bank shown below where the analysis filters are given by

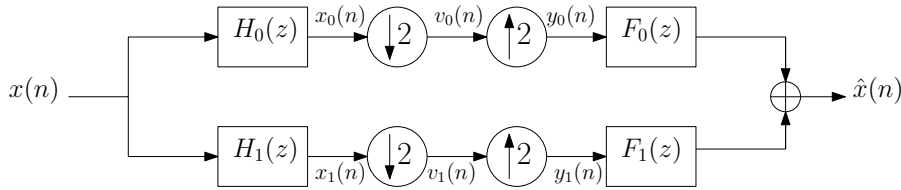


$$H_0(z) = 2 + 6z^{-1} + z^{-2} + 5z^{-3} + z^{-5}; H_1(z) = H_0(-z).$$

Find a set of stable synthesis filters that result in perfect reconstruction. (4 points)

**Solution:**

Let us label the signals as various junctions.



The signals at various nodes of the figure are

$$X_0(z) = H_0(z)X(z)$$

$$X_1(z) = H_1(z)X(z)$$

$$Y_0(z) = \frac{X_0(z) + X_0(-z)}{2}$$

$$Y_1(z) = \frac{X_1(z) + X_1(-z)}{2}$$

$$\hat{X}(z) = F_0(z)Y_0(z) + F_1(z)Y_1(z) = \frac{1}{2} [X_0(z)F_0(z) + X_1(z)F_1(z)] + \frac{1}{2} [X_0(-z)F_0(z) + X_1(-z)F_1(z)]$$

$$\implies \hat{X}(z) = \frac{1}{2} X(z) [H_0(z)F_0(z) + H_1(z)F_1(z)] + \frac{1}{2} X(-z) [H_0(-z)F_0(z) + H_1(-z)F_1(z)]$$

To force aliasing to zero,

$$[H_0(-z)F_0(z) + H_1(-z)F_1(z)] = 0 \quad \dots (1)$$

i.e.  $\frac{F_0(z)}{F_1(z)} = -\frac{H_1(-z)}{H_0(-z)}$

Then we have  $\hat{X}(z) = \frac{1}{2} [H_0(z)F_0(z) + H_1(z)F_1(z)] X(z)$

or  $\hat{X}(z) = T(z)X(z)$  where  $T(z) = \frac{1}{2} [H_0(z)F_0(z) + H_1(z)F_1(z)]$

For perfect reconstruction  $T(z) = cz^{-n_0}$

which implies  $\hat{x}(n) = cx(n - n_0)$  and

$$H_0(z)F_0(z) + H_1(z)F_1(z) = 2 \quad \dots (2)$$

Solving (1) and (2) by substituting  $H_0(z) = 2 + 6z^{-1} + z^{-2} + 5z^{-3} + z^{-5}$  and  $H_1(z) = H_0(-z)$  we get,

$$F_0(z) = \frac{2 + 6z^{-1} + z^{-2} + 5z^{-3} + z^{-5}}{24z^{-1} + 32z^{-3} + 14z^{-5} + 2z^{-7}}$$

$$F_1(z) = \frac{-2 + 6z^{-1} - z^{-2} + 5z^{-3} + z^{-5}}{24z^{-1} + 32z^{-3} + 14z^{-5} + 2z^{-7}}$$