Homework #2 solutions

Prayag
Linear and non-linear programming-1

March 20, 2018

Problem 1.

Solution. Let $x_1, \ldots, x_n$ be the set of vertices of the set P_F at which the optimal value of the LPP occurs i.e.,

$$C^{\mathrm{T}} x_i = m^\star \ \forall \ i = 1, \ldots, n \tag{1}$$

for some $m^\star < \infty$. Define x as a clc of $x_1, \ldots, x_n$ i.e.,

$$x = \sum_{i=1}^{n} \alpha_i x_i. \tag{2}$$

The cost at the point x is given by $C^{\mathrm{T}} x$ as

$$\begin{aligned} C^{\mathrm{T}} x &= C^{\mathrm{T}} \left(\sum_{i=1}^{n} \alpha_i x_i \right) \\ &= \sum_{i=1}^{n} \alpha_i C^{\mathrm{T}} x_i \\ &= m^\star \left(\sum_{i=1}^{n} \alpha_i \right) \\ &= m^\star \end{aligned}$$

■

Problem 2.

Solution. Let us consider the LPP

$$\begin{aligned} \text{minimize} \quad & C^{\mathrm{T}} x \\ \text{subject to} \quad & Ax = b \\ & x \geq 0. \end{aligned}$$

as L1. Since x_0 is an optimal solution of L1, we write $C^{\mathrm{T}} x_0 \leq C^{\mathrm{T}} x$ for any $x \in \Re^{\mathrm{n}}$. Hence, we get

$$C^{\mathrm{T}} x_0 \leq C^{\mathrm{T}} x^\star \tag{3}$$

. Similarly, let us call the LPP

$$\begin{aligned} \text{minimize} \quad & C^{\star\mathrm{T}} x \\ \text{subject to} \quad & Ax = b \\ & x \geq 0. \end{aligned}$$

as L2. Since, $x^\star$ is optimal solution of L2 we write $C^{\star\mathrm{T}} x^\star \leq C^{\star\mathrm{T}} x$ for any $x \in \Re^{\mathrm{n}}$. Hence, we write

$$C^{\star\mathrm{T}} x^\star \leq C^{\star\mathrm{T}} x_0. \tag{4}$$

Adding 3 and 4, we get $\left(C^{\mathrm{T}} - C^{\star\mathrm{T}}\right)(x^\star - x_0) \geq 0.$ ■

Problem 3.

Solution.

1. $Ad = 0$ and $Dd \leq 0 \implies d$ is feasible direction. Let $\theta > 0$ be a scalar and d be a vector in $\Re^{\mathrm{n}}$ space. For the vector d to be the feasible direction, the vector $x + \theta d$ should satisfy the following

$$\begin{aligned} A(x + \theta d) &= Ax + \theta Ad \\ &= b \end{aligned}$$

and

$$\begin{aligned} D(x + \theta d) &= Dx + \theta Dd \\ &\leq f - \theta(\delta)^2 \\ &\leq f \end{aligned}$$

for any $\delta \in \Re$. Therefore, vector d is a feasible direction.

2. d is a feasible direction $\implies Ad = 0$ and $Dd \leq 0$. Consider the following

$$\begin{aligned} A(x + \theta d) &= Ax + \theta Ad \\ & b + \theta Ad \end{aligned}$$

for $(b + \theta Ad) \in P$, we need $Ad = 0$. Similarly,

$$\begin{aligned} D(x + \theta d) &= Dx + \theta Dd \\ &= f + \theta Dd \end{aligned}$$

for $f + \theta Dd \in P$, we need $Dd = -(\delta)^2 \leq 0$ for any $\delta \in \Re$.

■

Problem 4.

Solution. (a) Let B_1 and B_2 be two different bases, let x_b be the basic solution. Since B_1 and B_2 leads to the same basic solution, we can write

$$B_1 x_b = b, \tag{5}$$
$$B_2 x_b = b. \tag{6}$$

Subtracting equations (5) and (6), we get

$$(B_1 - B_2)\, x_b = 0. \tag{7}$$

If every column of the matrix $(B_1 - B_2)$ is non zero and x_b nondegenerate, the columns of $(B_1 - B_2)$ are linearly dependent. Then the corresponding x_b can be made zero implying x_b has to be degenerate.

(b) Since rows of A are independent, the system $Bx_b = b$ has a unique solution. Where B is a matrix with linearly independent columns of A. Any degenerate x_b corresponds to only one basis and hence the answer is no.

(c) Note that two basic feasible solutions (vertices) are adjacent, if they use $m-1$ basic variables in common to form basis. Consider the following set of constraints

$$x_1 + x_2 = 1$$
$$x_2 + x_3 = 1.$$

The rank of matrix A is 2. Therefore, we get three bases $B_1 = \{x_1, x_2\}$, $B_2 = \{x_2, x_3\}$ and $B_3 = \{x_1, x_3\}$. The basic solution corresponding to B_1 and B_2 is $(0,1,0)^{\mathrm{T}}$ and $(1,0,1)^{\mathrm{T}}$ corresponding to B_3. We see that the two basic degenerate basic solutions are not adjacent to each other.

■

Problem 5.

Solution. Transform the given problem from maximization to minimization by multiplying the objective function by -1. With this transformation, convert the problem into standard form and follow the simplex tableau method. ■