Homework #1 solutions

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Linear and non-linear programming-1

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Problem 2.1.

Solution. (a) Introducing slack and surplus variables, we get

$$\begin{aligned} \text{minimize} \quad & x + 2y + z \\ \text{subject to} \quad & x + y - \alpha = 2 \\ & x + y + \beta = 3 \\ & x + z - \gamma = 4 \\ & x + z + \delta = 5 \\ & \alpha, \beta, \gamma, \delta \geq 0 \\ & x, y, z \geq 0. \end{aligned}$$

(b) Introducing slack and surplus variables, we get

$$\begin{aligned} \text{minimize} \quad & \alpha + \beta + \gamma + 4 \\ \text{subject to} \quad & \alpha + 2\beta + 3\gamma = 2 \\ & \alpha, \beta, \gamma \geq 0. \end{aligned}$$

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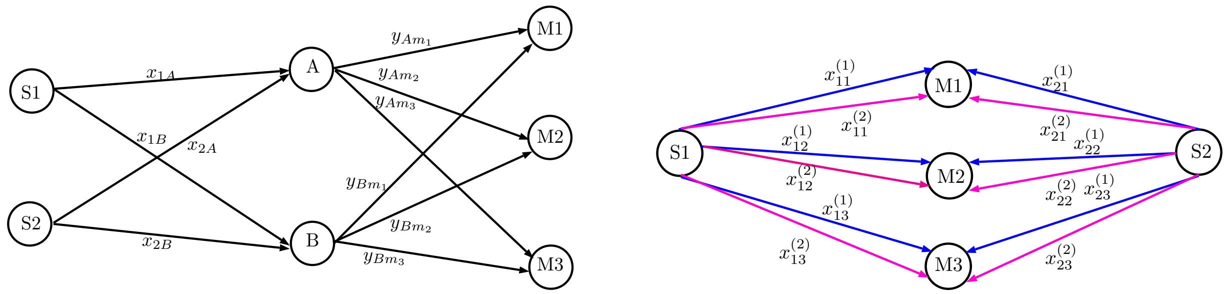
Problem 2.2.

Solution. (a) From Figure 1a, we see that the cost is given by

$$\begin{aligned} \text{Cost} = &(x_{1A} + 1.5x_{1B} + 2x_{2A} + 1.5x_{2B}) \\ &+ (y_{Am_1} + 2y_{Am_2} + y_{Am_3} + 3y_{Bm_1} + 4y_{Bm_2} + 2y_{Bm_3}) \end{aligned} \tag{1}$$



(a) (b)

Figure 1

$$
\begin{aligned}
\text{minimize} \quad & \text{Cost (Equation 1)} \\
\text{subject to} \quad & x_{1A} + x_{1B} && = 10 \\
& x_{2A} + x_{2B} && = 15 \\
& y_{Am_1} + y_{Bm_1} && = 8 \\
& y_{Am_2} + y_{Bm_2} && = 14 \\
& y_{Am_3} + y_{Bm_3} && = 3 \\
& x_{1A}, x_{1B}, x_{2A}, x_{2B} && \geq 0 \\
& y_{Am_1}, y_{Am_2}, y_{Am_3} && \geq 0 \\
& y_{Bm_1}, y_{Bm_2}, y_{Bm_3} && \geq 0.
\end{aligned}
$$

(b) From Figure 1b, we see that the cost is given by

$$
\begin{aligned}
\text{Cost} = & \left(5x_{11}^{(1)} + 3.5x_{11}^{(2)} + 3x_{12}^{(1)} + 5.5x_{12}^{(2)} + 2x_{13}^{(1)} + 3.5x_{13}^{(2)}\right) \\
& + \left(6x_{21}^{(1)} + 4.5x_{21}^{(2)} + 3x_{22}^{(1)} + 3.5x_{22}^{(2)} + 2x_{23}^{(1)} + 3.5x_{23}^{(2)}\right)
\end{aligned} \tag{2}
$$

$$
\begin{aligned}
\text{minimize} \quad & \text{Cost (Equation 2)} \\
\text{subject to} \quad & \sum_{i=1}^{2} x_{1k}^{(i)} + x_{2k}^{(i)} = \alpha_k \ \forall k = 1, 2, 3 \\
& \sum_{k=1}^{3} x_{1k}^{(1)} + x_{1k}^{(2)} = 10 \\
& \sum_{k=1}^{3} x_{2k}^{(1)} + x_{2k}^{(2)} = 15 \\
& x_{1k}^{(1)}, x_{2k}^{(1)} \geq 0 \ \forall \ k = 1, 2, 3.
\end{aligned}
$$

where

$$
\alpha_k = \begin{cases} 8 & k = 1 \\ 14 & k = 2 \\ 3 & k = 3 \end{cases}
$$

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Problem 2.8.

Solution. Let us assume the following: $|x| \leq t_1,\ |y| \leq t_2,\ |z| \leq t_3$ for some $t_1, t_2,$ and $t_3 \geq 0$. This results in the following inequalities.

$$x \leq t_1 \tag{3}$$
$$x \geq -t_1 \tag{4}$$
$$y \leq t_2 \tag{5}$$
$$y \geq -t_2 \tag{6}$$
$$z \leq t_3 \tag{7}$$
$$z \geq -t_3 \tag{8}$$

Introducing slack and surplus variables we get

$$2x + z = 3 \tag{9}$$
$$x + y + \alpha_1 = 1 \tag{10}$$
$$x + \alpha_2 = t_1 \tag{11}$$
$$x - \alpha_3 = -t_1 \tag{12}$$
$$y + \alpha_4 = t_2 \tag{13}$$
$$y - \alpha_5 = -t_2 \tag{14}$$
$$z + \alpha_6 = t_3 \tag{15}$$
$$z - \alpha_7 = -t_3 \tag{16}$$

From equations (9)-(16), we get

$$2t_1 - \alpha_2 - \alpha_3 = 0 \tag{17}$$
$$2t_2 - \alpha_4 - \alpha_5 = 0 \tag{18}$$
$$2t_3 - \alpha_6 - \alpha_7 = 0 \tag{19}$$
$$t_1 + t_2 - \alpha_1 - \alpha_3 - \alpha_5 = -1 \tag{20}$$
$$2t_1 + t_3 - \alpha_1 - 2\alpha_3 = -3 \tag{21}$$

The standard form the problem is given by

$$\begin{aligned} \text{minimize} \quad & t_1 + t_2 + t_3 \\ \text{subject to} \quad & t_1 + t_2 - \alpha_1 - \alpha_3 - \alpha_5 = -1 \\ & 2t_1 + t_3 - \alpha_1 - 2\alpha_3 = -3 \\ & 2t_1 - \alpha_2 - \alpha_3 = 0 \\ & 2t_2 - \alpha_4 - \alpha_5 = 0 \\ & 2t_3 - \alpha_6 - \alpha_7 = 0 \\ & t_i, \alpha_j \geq 0 \ \forall\ i = 1, 2, 3 \text{ and } j = 1, \ldots, 7. \end{aligned}$$

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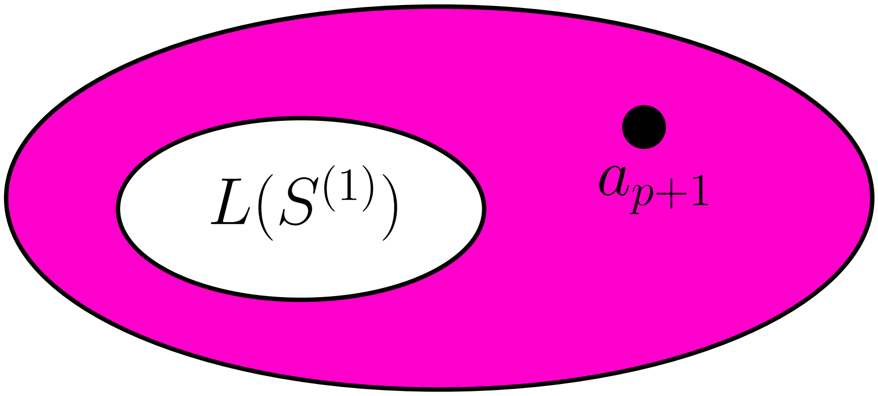

Figure 2

Problem 2.12.

Solution. Let $S^{(1)} = \{a_1, \dots, a_p\}$ and $L(S^{(1)}) = \text{span}\{a_1, \dots, a_p\}$ denote the span of vectors as shown in Figure 2. Consider a vector $a_{p+1} \notin L(S^{(1)})$. Now, equating linear combination of $\{a_1, \dots, a_{p+1}\}$ to zero we get

$$\alpha_1 a_1 + \cdots + \alpha_p a_p + \alpha_{p+1} a_{p+1} = 0. \tag{22}$$

From equation (22) we see that $a_{p+1} = 0$ (because the set $S^{(1)}$ is a linearly independent set). Now, the set $S^{(2)} = \{a_1, \dots, a_p, a_{p+1}\}$ forms a linearly independent set. Similarly consider another vector $a_{p+2} \notin L(S^{(2)})$ and we see that set $S^{(3)} = \{a_1, \dots, a_{p+1}, a_{p+2}\}$ forms a linearly independent set. This can be repeated till we find $m - p$ vectors.

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Problem 2.3.

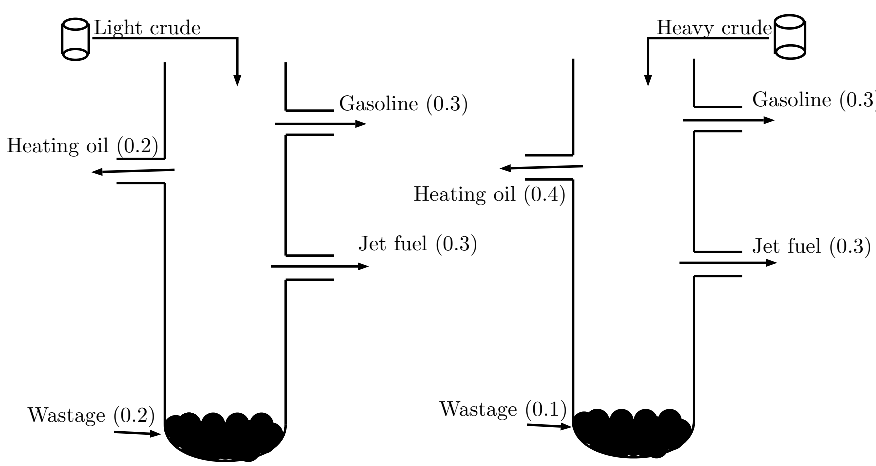

Figure 3

Solution. From the table we see that, 1 barrel of light crude produces 0.3 barrels of gasoline, 0.2 barrels of heating oil, and 0.3 barrels of jet fuel. Similarly, 1 barrel of heavy crude produces 0.3 barrels of gasoline, 0.4 barrels of heating oil, and 0.2 barrels of jet fuel. Let x_l and x_h be the barrels of light crude and heavy crude respectively. Therefore, the cost is

$$C = 35x_l + 30x_h \tag{23}$$

The standard form of the problem is

$$\begin{aligned} \text{minimize} \quad & C \\ \text{subject to} \quad & 0.3x_l + 0.3x_h = 900000 \\ & 0.2x_l + 0.4x_h = 800000 \\ & 0.3x_l + 0.2x_h = 500000 \\ & x_l, x_h \geq 0. \end{aligned}$$

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Problem 2.9.

Solution. Let $y = \text{maximum}\left(c_1^{\mathrm{T}}x + d_1, \ldots, c_p^{\mathrm{T}}x + d_p\right)$. Then we have to minimize y subject to the following constraints:

$$\begin{aligned} \text{minimize} \quad & y \\ \text{subject to} \quad & c_i^{\mathrm{T}}x + d_i \leq y \\ & c_i, d_i \geq 0 \ \forall\ i = 1, \ldots, p. \end{aligned}$$

Introducing slack variables we get

$$\begin{aligned} \text{minimize} \quad & y \\ \text{subject to} \quad & c_i^{\mathrm{T}}x + d_i + \alpha_i = y \\ & c_i, d_i, \alpha_i \geq 0 \ \forall\ i = 1, \ldots, p. \end{aligned}$$

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