

Indian Institute of Science  
E9–207: Basics of Signal Processing  
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Homework #1 Solutions, Spring 2018  
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**PROBLEM 1:**

Give examples (if any) of 2D discrete time systems that are (a) non-causal but stable (b) non-causal and unstable. (4 pts.)

**Solution:**

(a) Non-causal but stable:

$$y(n_1, n_2) = x(n_1 + k_1, n_2 + k_2)$$

$n_1$  and  $n_2$  are time parameters in 2 dimensions and  $k_1$  and  $k_2$  are positive integers. The system is non-causal because output depends on future values of input. The system is stable because output is bounded if input is bounded.

(b) Non-causal and unstable:

$$y(n_1, n_2) = \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} x(n_1 + k_1, n_2 + k_2)$$

The system is non-causal because output depends on future values of input. The system is unstable because output is unbounded for bounded input.

**PROBLEM 2:**

Find the modes of a system whose impulse response is  $y[n] = n^k u[n]$ , where  $k$  is a positive integer. Write the governing difference equation for this system. (6 pts.)

**Solution:**

The impulse response is given as  $y[n] = n^k u[n]$ .

Let  $Y(z)$  denote the z-transform of the system.

The Z-transform of a unit step function is,

$$\sum_{n=-\infty}^{\infty} u[n]z^{-n} = \frac{1}{1 - z^{-1}} = \frac{z}{z - 1}$$

Differentiating w.r.t.  $z$  on both sides we have,

$$\begin{aligned} \sum_{n=-\infty}^{\infty} nu[n]z^{-n-1} &= \frac{1}{(z - 1)^2} \\ \implies \sum_{n=-\infty}^{\infty} nu[n]z^{-n} &= \frac{z}{(z - 1)^2} \end{aligned}$$

Repeating the differentiation  $k-1$  more times we get,

$$Y(z) = n^k u[n]z^{-n} = \frac{f(z)}{(z - 1)^{k+1}}$$

for some polynomial  $f(z)$ .

Therefore,  $z = 1$  is  $(k + 1)^{\text{th}}$  order pole. Therefore the system has  $k + 1$  modes.

Taking Z-transform of  $u[n]$  we get,

$$u[n] \rightarrow \frac{1}{1 - z^{-1}}$$

Similarly,

$$n u[n] \rightarrow \frac{z}{(z - 1)^2}$$

and

$$n^2 u[n] \rightarrow \frac{z(z + 1)}{(z - 1)^3}$$

Generalizing this for  $k^{\text{th}}$  power of  $n$ , we get,

$$n^k u[n] \rightarrow \frac{1}{(z - 1)^{k+1}} \sum_{n=0}^{k-1} \left\langle \begin{matrix} k \\ n \end{matrix} \right\rangle z^{n+1}$$

where  $\left\langle \begin{matrix} k \\ n \end{matrix} \right\rangle$  for  $n = 0$  to  $n = k - 1$  are Eulerian numbers.

So,

$$\frac{Y(z)}{X(z)} = \frac{1}{(1 - z^{-1})^{k+1}} \sum_{n=0}^{k-1} \left\langle \begin{matrix} k \\ n \end{matrix} \right\rangle z^{n-k}$$

$$\Rightarrow (1 - z^{-1})^{k+1} Y(z) = \sum_{n=0}^{k-1} \left\langle \begin{matrix} k \\ n \end{matrix} \right\rangle z^{n-k} X(z)$$

$$\begin{aligned} \Rightarrow & \binom{k+1}{0} Y(z) - \binom{k+1}{1} z^{-1} Y(z) + \binom{k+1}{2} z^{-2} Y(z) - \binom{k+1}{3} z^{-3} Y(z) + \dots + (-1)^{k+1} \binom{k+1}{k+1} z^{-(k+1)} Y(z) \\ & = \left\langle \begin{matrix} k \\ 0 \end{matrix} \right\rangle z^{-k} X(z) + \left\langle \begin{matrix} k \\ 1 \end{matrix} \right\rangle z^{-(k-1)} X(z) + \left\langle \begin{matrix} k \\ 2 \end{matrix} \right\rangle z^{-(k-2)} X(z) + \dots + \left\langle \begin{matrix} k \\ k-1 \end{matrix} \right\rangle z^{-1} X(z) \end{aligned}$$

Thus we get difference equation of the system as,

$$\begin{aligned} & \binom{k+1}{0} y[n] - \binom{k+1}{1} y[n-1] + \binom{k+1}{2} y[n-2] - \binom{k+1}{3} y[n-3] + \dots + (-1)^{k+1} y[n - (k+1)] \\ & = \left\langle \begin{matrix} k \\ 0 \end{matrix} \right\rangle x[n-k] + \left\langle \begin{matrix} k \\ 1 \end{matrix} \right\rangle x[n - (k-1)] + \left\langle \begin{matrix} k \\ 2 \end{matrix} \right\rangle x[n - (k-2)] + \dots + \left\langle \begin{matrix} k \\ k-1 \end{matrix} \right\rangle x[n-1] \\ & \Rightarrow \sum_{i=0}^{k+1} (-1)^i \binom{k+1}{i} y[n-i] = \sum_{j=0}^{k-1} \left\langle \begin{matrix} k \\ j \end{matrix} \right\rangle x[n - (k-j)] \end{aligned}$$

### PROBLEM 3:

Three coins  $C_1$ ,  $C_2$  and  $C_3$  each show heads with probability  $3/5$  and tails otherwise.  $C_1$  counts 10 points for a head

and 2 for a tail,  $C_2$  counts 4 points for both head and tail and  $C_3$  counts 3 points for a head and 20 for a tail. You and your opponent each choose a coin; you cannot choose the same coin as your opponent. Each of you tosses your coin and the person with the larger score wins 10 dollars. Would you prefer to be the first to pick a coin or the second? Explain. (5 pts.)

**Solution:**

It may appear intuitively that the person who goes first has more chances of winning. But this is not true. Let us discuss different cases.

**Case 1:** Your opponent chooses  $C_1$ . In this case, you should choose  $C_3$ .

**Reason:** Probability that  $C_3$  beats  $C_1 = P(\text{HT or TH or TT}) = P(\text{HT}) + P(\text{TH}) + P(\text{TT}) = \frac{6+6+4}{25} = \frac{16}{25}$ .

**Case 2:** Your opponent chooses  $C_2$ . In this case, you should choose  $C_1$ .

**Reason:** Probability that  $C_1$  beats  $C_2 = P(\text{HH or TH}) = P(\text{HH}) + P(\text{TH}) = \frac{9+6}{25} = \frac{3}{5}$ .

**Case 3:** Your opponent chooses  $C_3$ . In this case, you should choose  $C_2$ .

**Reason:** Probability that  $C_2$  beats  $C_3 = P(\text{HH or HT}) = P(\text{HH}) + P(\text{HT}) = \frac{9+6}{25} = \frac{3}{5}$ .

No matter which coin your opponent chooses, you can choose a coin which has more probability of winning against your opponent's coin. Thus, you would prefer to let your opponent have the first go.

**PROBLEM 4:**

Three companies  $A, B, C$  manufacture light bulbs and have a market share in the ratio  $0.35 : 0.35 : 0.3$ . Probability of each of them producing a defective bulb is respectively  $0.01, 0.02$  and  $0.05$ . A randomly chosen bulb is found defective. What is the probability that it was manufactured by company  $B$ ? (4 pts.)

**Solution:**

Let  $P(A), P(B)$  and  $P(C)$  denote the probabilities that a bulb has been manufactured by company  $A, B$  and  $C$  respectively. Let  $P(D)$  denote the probability that a certain bulb is defective.

We know,  $P(D|A) = 0.01, P(D|B) = 0.02$  and  $P(D|C) = 0.05$ . We need to evaluate  $P(B|D)$ .

Applying Bayes' theorem

$$\begin{aligned} P(B|D) &= \frac{P(D|B) \times P(B)}{P(D|A) \times P(A) + P(D|B) \times P(B) + P(D|C) \times P(C)} \\ &= \frac{0.02 \times 0.35}{0.01 \times 0.35 + 0.02 \times 0.35 + 0.05 \times 0.3} \\ &= \frac{14}{51} \end{aligned}$$

Thus if a randomly chosen bulb is found defective, probability that it was manufactured by company  $B$  is  $\frac{14}{51}$ .

**PROBLEM 5:**

(a) If  $A$  and  $B$  are independent events. Show that  $A^c$  and  $B^c$  are independent. (3 pts.)

(b) If  $A$  and  $B$  are two events in  $\Omega$ , show that (i)  $(A \cup B)^c = A^c \cap B^c$  (ii)  $(A \cap B)^c = A^c \cup B^c$  (4 pts.)

**Solution:**

(a)

$$\begin{aligned} P(A^c \cap B^c) &= 1 - P(A \cup B) \\ &= 1 - P(A) - P(B) + P(A)P(B) \\ &= P(A^c) - P(B)(1 - P(A)) \\ &= P(A^c) - P(B)P(A^c) \\ &= P(A^c)P(B^c) \end{aligned}$$

(b) We show  $(A \cup B)^c \subseteq A^c \cap B^c$  and  $A^c \cap B^c \subseteq (A \cup B)^c$

Let  $x \in (A \cup B)^c \Rightarrow x \notin (A \cup B) \Rightarrow x \notin A$  and  $x \notin B \Rightarrow x \in A^c$  and  $x \in B^c \Rightarrow x \in A^c \cap B^c$   
 $(A \cup B)^c \subseteq A^c \cap B^c$

Let  $x \in A^c \cap B^c \Rightarrow x \in A^c$  and  $x \in B^c \Rightarrow x \notin A$  and  $x \notin B \Rightarrow x \notin A \cup B \Rightarrow x \in (A \cup B)^c$   
 $A^c \cap B^c \subseteq (A \cup B)^c$   
 $\therefore (A \cup B)^c = A^c \cap B^c$

We show  $(A \cap B)^c \subseteq A^c \cup B^c$  and  $A^c \cup B^c \subseteq (A \cap B)^c$

Let  $x \in (A \cap B)^c \Rightarrow x \notin (A \cap B) \Rightarrow x \notin A$  or  $x \notin B \Rightarrow x \in A^c$  or  $x \in B^c \Rightarrow x \in A^c \cup B^c$   
 $(A \cap B)^c \subseteq A^c \cup B^c$

Let  $x \in A^c \cup B^c \Rightarrow x \in A^c$  or  $x \in B^c \Rightarrow x \notin A$  or  $x \notin B \Rightarrow x \notin A \cap B \Rightarrow x \in (A \cap B)^c$   
 $A^c \cup B^c \subseteq (A \cap B)^c$   
 $\therefore (A \cap B)^c = A^c \cup B^c$

**PROBLEM 6:**

If  $X$  is a continuous random variable with pdf given by  $f(t) = \frac{1}{\pi(1+t^2)}$ , for  $-\infty < t < \infty$ . What can you say about the mean and variance of  $X$ ? (6 pts.)

**Solution:**

This is an example of a random variable having Cauchy distribution.

$$\begin{aligned} \mathbb{E}[X] &= \int_{-\infty}^{\infty} t \frac{1}{\pi(1+t^2)} dt \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} 2t \frac{1}{(1+t^2)} dt \\ &= \frac{1}{2\pi} \ln(1+t^2) \Big|_{-\infty}^{\infty} \end{aligned}$$

This is undefined.

Similarly, variance  $\sigma^2 = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$

Compute  $\mathbb{E}[X^2]$

$$\begin{aligned} \mathbb{E}[X^2] &= \int_{-\infty}^{\infty} t^2 \frac{1}{\pi(1+t^2)} dt \\ &= \int_{-\infty}^{\infty} \frac{1+t^2-1}{\pi(1+t^2)} dt \\ &= \frac{1}{\pi} \int_{-\infty}^{\infty} \left(1 - \frac{1}{(1+t^2)}\right) dt \\ &= \frac{1}{\pi} \left( t \Big|_{-\infty}^{\infty} - \tan^{-1} t \Big|_{-\infty}^{\infty} \right) \end{aligned}$$

This is also undefined. Therefore, both the mean and variance are undefined.

**PROBLEM 7:**

Consider 2D points lying on a circular disk of radius  $R$  centered at origin.

$$S := \{(x, y) : x^2 + y^2 \leq R^2\}$$

All points  $(x, y) \in S$  are uniformly distributed on the disk. Obtain the marginal distributions  $f_X(x)$  and  $f_Y(y)$ . Are they statistically independent? Are they correlated? (8 pts.)

**Solution:**

Joint pdf of random variables  $X$  and  $Y$  uniformly distributed in  $S$  can be written as

$$f_{X,Y}(x, y) = \begin{cases} \frac{1}{\pi R^2} & \text{for } (x, y) \in S \text{ i.e., } x^2 + y^2 \leq R^2 \\ 0 & \text{otherwise} \end{cases}$$

The marginal probability density for random variable  $X$  can be evaluated as

$$\begin{aligned} f_X(x) &= \int_{-\infty}^{\infty} f_{X,Y}(x,y) dy \\ &= \int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} \frac{1}{\pi R^2} dy \quad \left( \text{probability of points outside } S \text{ is zero} \right) \\ &= \frac{2}{\pi R^2} \sqrt{R^2 - x^2} \end{aligned}$$

Therefore, including points of measure 0, we have

$$f_X(x) = \begin{cases} \frac{2}{\pi R^2} \sqrt{R^2 - x^2} & \text{for } -R \leq x \leq R \\ 0 & \text{otherwise} \end{cases}$$

Similarly, the marginal probability density for random variable  $Y$  can be evaluated as

$$\begin{aligned} f_Y(y) &= \int_{-\infty}^{\infty} f_{X,Y}(x,y) dx \\ &= \int_{-\sqrt{R^2-y^2}}^{\sqrt{R^2-y^2}} \frac{1}{\pi R^2} dx \quad \left( \text{probability of points outside } S \text{ is zero} \right) \\ &= \frac{2}{\pi R^2} \sqrt{R^2 - y^2} \end{aligned}$$

Therefore, including points of measure 0, we have

$$f_Y(y) = \begin{cases} \frac{2}{\pi R^2} \sqrt{R^2 - y^2} & \text{for } -R \leq y \leq R \\ 0 & \text{otherwise} \end{cases}$$

Since  $f_{XY}(x,y) \neq f_X(x)f_Y(y)$ , we infer that  $X$  and  $Y$  are not independent.

To check for correlation, we evaluate  $\mathbb{E}[XY]$

$$\begin{aligned} \mathbb{E}[XY] &= \int_x \int_y xy f_{XY}(x,y) dx dy \\ &= \int_{-R}^R x \left( \int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} \frac{y}{\pi R^2} dy \right) dx \\ &= 0 \quad \left( \int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} \frac{y}{\pi R^2} dy = 0 \text{ since integral of odd function} \right) \\ \mathbb{E}[X] &= \int_{-R}^R x \left( \frac{2\sqrt{R^2-x^2}}{\pi R^2} \right) dx = 0 \quad \left( \text{integral of odd function} \right) \\ \mathbb{E}[Y] &= \int_{-R}^R y \left( \frac{2\sqrt{R^2-y^2}}{\pi R^2} \right) dy = 0 \quad \left( \text{integral of odd function} \right) \end{aligned}$$

Covariance  $\text{Cov}(X, Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y] = 0$ . Therefore,  $X$  and  $Y$  are uncorrelated.

**PROBLEM 8:**

Consider two uniformly distributed  $U[0, 1]$  random variables  $X$  and  $Y$ . Let us form a new random variable  $Z = |X - Y|$ . Find and sketch (a) Probability distribution function of  $Z$  (b) Probability density function of  $Z$ . Also compute the mean and variance of  $Z$ . (10 pts.)

**Solution:**

(a) Probability distribution function of  $Z$ :

$F_Z(z) = P(Z \leq z)$ . We note that as  $Z = |X - Y|$  for  $z \geq 1$ ,  $|X - Y| \leq z$  with probability 1. We evaluate below the

distribution function for  $z \leq 1$ .

$$\begin{aligned}
 F_Z(z) &= P(Z \leq z) = P(|X - Y| \leq z) \\
 &= P(-z \leq X - Y \leq z) \\
 &= P(\{-z \leq X - Y\} \cap \{X - Y \leq z\}) \\
 &= 1 - P(\{-z \leq X - Y\}^c \cup \{X - Y \leq z\}^c) \\
 &= 1 - P(\{-z \leq X - Y\}^c) - P(\{X - Y \leq z\}^c) + P(\{-z \leq X - Y\}^c \cap \{X - Y \leq z\}^c) \\
 &= 1 - P(\{X - Y < -z\}) - P(\{X - Y > z\}) + P(\{X - Y < -z\} \cap \{X - Y > z\}) \\
 &= 1 - P(\{X - Y < -z\}) - P(\{X - Y > z\})
 \end{aligned}$$

Let us evaluate  $P(\{X - Y < -z\})$

$$\begin{aligned}
 P(\{X - Y < -z\}) &= \int_{y=z}^1 \int_{x=0}^{y-z} f_{XY}(x, y) dy dx \\
 &= \int_{y=z}^1 \int_{x=0}^{y-z} 1 \cdot dx dy \\
 &= \int_{x=z}^1 (y - z) dy = \frac{1}{2} - z - z^2
 \end{aligned}$$

Let us evaluate  $P(\{X - Y > z\})$

$$\begin{aligned}
 P(\{X - Y > z\}) &= \int_{x=z}^1 \int_{y=0}^{x-z} f_{XY}(x, y) dy dx \\
 &= \int_{x=z}^1 \int_{y=0}^{x-z} 1 \cdot dy dx \\
 &= \int_{x=z}^1 (x - z) dx = \frac{1}{2} - z - z^2
 \end{aligned}$$

Therefore,  $F_Z(z) = 1 - P(X - Y > z) - P(X - Y < -z) = 2z - z^2$  for  $0 \leq z \leq 1$ .

$$F_Z(z) = \begin{cases} 0 & \text{for } z < 0 \\ 2z - z^2 & \text{for } 0 \leq z \leq 1 \\ 1 & \text{for } z \geq 1 \end{cases} \quad (1)$$

(b) Probability density function of  $Z$ :

$$f_Z(z) = \frac{d}{dz} F_Z(z) = \begin{cases} 2 - 2z & \text{for } 0 \leq z \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Computation of mean and variance:

$$\begin{aligned}
 \mathbb{E}[Z] &= \int_z z f_Z(z) dz \\
 &= \int_{z=0}^1 z(2 - 2z) dz = \frac{1}{3} \\
 \mathbb{E}[Z^2] &= \int_z z^2 f_Z(z) dz \\
 &= \int_{z=0}^1 z^2(2 - 2z) dz = \frac{1}{6} \\
 \text{variance} &= \mathbb{E}[Z^2] - (\mathbb{E}[Z])^2 \\
 &= \frac{1}{6} - \frac{1}{9} = \frac{1}{18}
 \end{aligned}$$