Unique Representation Theorem

Theorem 1. *Let S be a vector space and $T \subset S$ be non empty. The set T is linearly independent iff for each non-zero $\underline{x} \in \text{span}(T)$, there is exactly one finite subset of T denoted by $\{\underline{p_1}, \underline{p_2}, \ldots, \underline{p_n}\}$ and unique set of scalars $c_1, c_2, \ldots, c_n$ such that,*

$$\underline{x} = c_1\underline{p_1} + c_2\underline{p_2} + \cdots + c_n\underline{p_n} \tag{1}$$

Proof.

Linear independence $\Rightarrow$ Unique Representation

We prove this by contradiction. Let T be a linearly independent set. Let us assume that there exists $\underline{x} \in \text{span}(T)$ whose representation using T is not unique. Thus, there exists two subsets of T, namely $P = \{\underline{p_1}, \underline{p_2}, \ldots, \underline{p_m}\}$ and $Q = \{\underline{q_1}, \underline{q_2}, \ldots, \underline{q_n}\}$ such that,

$$\underline{x} = \sum_{i=1}^{m} c_i\underline{p_i} = \sum_{i=1}^{n} d_i\underline{q_i}$$

where c_i's and d_i's are non-zero. Rearranging the terms, we obtain,

$$\sum_{i=1}^{m} c_i\underline{p_i} - \sum_{i=1}^{n} d_i\underline{q_i} = \underline{0} \tag{2}$$

As $\underline{p_i}$'s and $\underline{q_i}$'s belong to T, if $P \cap Q = \emptyset$ then all $\underline{p_i}$'s and $\underline{q_i}$'s are different. This contradicts the fact that T is a linearly independent set as their non trivial linear combination cannot sum to $\underline{0}$. Hence, there must be some overlap between the two sets.

Let $m < n$. Equation 2 holds only if for every $\underline{p_i}$, there exists some $\underline{q_j}$ such that $\underline{p_i} = \underline{q_j}$ and $c_i - d_j = 0$. This is true as only trivial linear combination of the vectors in T can be $\underline{0}$. Renumbering the elements in Q, we obtain

$$\underline{p_i} = \underline{q_i} \text{ and } c_i = d_i. \tag{3}$$

Thus, $P \subset Q$. From equation 2 and 3,

$$\sum_{i=m+1}^{n} d_i\underline{q_i} = \underline{0} \tag{4}$$

As, if $\underline{q_i}$'s are nonzero they should be linearly independent and d_i are non-zero, the only possible solution is $\underline{q_i} = \underline{0}$. Neglecting the zero vector, we redefine $Q = \{\underline{q_1}, \underline{q_2}, \ldots, \underline{q_m}\} = P$. Thus, the representation is unique.

Unique Representation $\Rightarrow$ Linear independence

We prove this by contradiction. Let every vector $\underline{x} \in \text{span}(T)$ have a unique representation in terms of vectors in $T = \{\underline{t_1}, \underline{t_2}, \ldots, \underline{t_k}\}$. Let us assume that T is a linearly dependent set, then there exists $a_1, a_2, \ldots, a_k$, where atleast one a_i is non-zero, such that,

$$\sum_{i=1}^{k} a_i\underline{t_i} = 0. \tag{5}$$

Let a_1 be non-zero. Consider $\underline{x} = \underline{t_1} \in \text{span}(T)$. From equation 5,

$$\underline{x} = \underline{t_1} = -\frac{1}{a_1}\sum_{i=2}^{k} a_i\underline{t_i}. \tag{6}$$

As $\underline{x}$ doesnot have a unique representation, this leads to a contradiction. Hence, T is a linearly independent set. $\square$