Indian Institute of Science
E9−252: Mathematical Methods and Techniques in Signal Processing
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Homework #5 Solutions, Fall 2017
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Late submission policy: Points scored = Correct points scored × e^{-d}, d = # days late
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PROBLEM 1:
Problem 5.15 from P.P Vaidyanathan's Book

Solution:

$$\begin{aligned}
X_k(z) &= \frac{1}{M}\sum_{i=0}^{M-1} z^{-\frac{kJ}{M}} W^{-ikJ} X(z^{\frac{1}{M}} W^i), \qquad 0 \le k \le M-1 \\
&= \frac{z^{\frac{-kJ}{M}}}{M} \sum_{i=0}^{M-1} W^{-ikJ} X(z^{\frac{1}{M}} W^i), \\
y_k(z) &= \frac{z^{\frac{-kJ}{M}}}{M} \sum_{i=0}^{M-1} W^{-ikJ} X(zW^i), \qquad 0 \le k \le M-1 \\
\widehat{X}(z) &= \sum_{k=0}^{M-1} z^{-(M-1-k)J} \frac{z^{-kJ}}{M} \sum_{i=0}^{M-1} W^{-ikJ} X(zW^i), \\
&= \sum_{k=0}^{M-1} \frac{z^{-(M-1)J}}{M} \sum_{i=0}^{M-1} W^{-ikJ} X(zW^i), \\
&= \frac{z^{-(M-1)J}}{M} \sum_{i=0}^{M-1} X(zW^i) \sum_{K=0}^{M-1} W^{-ikJ}.
\end{aligned}$$

When $i = 0$, $\sum_{k=0}^{M-1} W^{-ikJ} = \sum_{k=0}^{M-1} 1 = M$. When $i \neq 0$, $\sum_{k=0}^{M-1} W^{-ikJ} = \sum_{k=0}^{M-1} e^{\frac{j2\pi kiJ}{M}}$

If iJ is a multiple of M, $\sum_{k=0}^{M-1} W^{-ikJ} = \sum_{k=0}^{M-1} 1 = M$.

If iJ is not a multiple of M, $\sum_{k=0}^{M-1} W^{-ikJ} = \sum_{k=0}^{M-1} e^{\frac{j2\pi kiJ}{M}} = \frac{1 - e^{\frac{j2\pi iJk}{M}M}}{1 - e^{\frac{j2\pi iJ}{M}}} = \frac{0}{1 - e^{\frac{ji2\pi J}{M}}} = 0$

Note that the denominator doesn't go to 0 as iJ is not a multiple of M.

If J and M are relatively prime, iJ cannot be a multiple of M as $i < M$ always.

Thus, when J and M are relatively prime, $\sum_{k=0}^{M-1} W^{-ikJ} = 0, \forall i \neq 0$.

$$\begin{aligned}
\Longrightarrow \widehat{X}(z) &= \frac{z^{-(M-1)J}}{M} X(z) M = z^{-(M-1)J} X(z) \\
\Longrightarrow \widehat{X}(n) &= x(n-(M-1)J) \qquad \text{(we obtain perfect reconstruction)}
\end{aligned}$$

If J and M are not relatively prime, then let $g = \gcd(M, J)$

Choose $i = \frac{M}{\gcd(M,J)}$. Observe $i < M$. Thus, $iJ = \frac{M}{\gcd(M,J)}\left(\frac{J}{\gcd(M,J)}\gcd(M,J)\right) = M\frac{J}{\gcd(M,J)}$. This is a multiple of M as $\frac{J}{\gcd(M,J)} \in \mathbb{N}$

$\therefore \widehat{X}(z)$ has atleast one more term other than $\frac{z^{-(M-1)J}}{M}X(z)M$, that is,

$$\widehat{X}(z) = z^{-(M-1)J}X(z) + z^{-(M-1)J}X(zW^{\frac{M}{\gcd(M,J)}}) + \text{other terms}$$

Thus, $\widehat{X}(z)$ has aliasing components and hence perfect reconstruction cannot be obtained as $\widehat{x}(n)$ would not be a scaled and time shifted version of $x(n)$.

Thus, perfect reconstruction is achieved iff M and J are relatively prime.

PROBLEM 2:
Problem 5.18 from P.P Vaidyanathan's Book

Solution:

Now, as the choice of filters are such that there is perfect reconstruction, thus,

$$\begin{aligned} \frac{1}{M}\sum_{k=0}^{M-1} H_k(zW^g)F_k(z) &= 0, \quad 1 \leq g \leq M-1 \quad \text{(no aliasing)} \\ \text{and } \frac{1}{M}\sum_{k=0}^{M-1} H_k(z)F_k(z) &= cz^{-n_0} \quad \text{(PR property)} \end{aligned}$$

If we replace $F_k(z)$ by $F_k(zW^l)$ then,

$\widehat{X_1}(z) = \frac{1}{M}\sum_{g=0}^{M-1} X(zW^g)\sum_{k=0}^{M-1} H_k(zW^g)F_k(zW)$

Let us define $G_i(z) = \frac{1}{M}\sum_{k=0}^{M-1} H_k(zW^i)F_k(z), \quad 0 \leq i \leq M-1$

If $l = 0$, then there is no change $\Rightarrow \widehat{X}_1(n) = \widehat{X}(n) = c_x(n - n_0) \Rightarrow$ Perfect reconstruction.

If $l \neq 0$, then for $1 \leq l \leq M-1$

$$\frac{1}{M}\sum_{k=0}^{M-1} H_k(z)F_k(zW^l) = G_{m-1}(zW^l) = 0 \text{ as } l \neq M \text{ and } l \neq 0$$

For $1 \leq g \leq M-1$,

$$\frac{1}{M}\sum_{k=0}^{M-1} H_k(zW^g)F_k(zW^l) = G_{(M-l+g) \bmod M}(zW^l) = \begin{cases} 0 & M-l+g \bmod M \neq 0 \\ cz^{-2n_0} & M-l+g \bmod M = 0 \end{cases}$$

the condition $M - l + g \bmod M = 0$ is satisfied if g=l. therefore $\widehat{X_1}(z) = x(zW)c(zW^l)^{-n_0}$

$$\Longrightarrow \widehat{X_1}(n) = ce^{\frac{j2\pi ln}{M}x(n-n_0)}$$

as the system doesn't satisfy the perfect reconstruction property , we cannot recover x(n).

PROBLEM 3:
Problem 5.33 from P.P Vaidyanathan's Book

Solution: Suppose the system has PR property, then

$$\begin{aligned}\frac{1}{M}\sum_{k=0}^{M-1} H_k(z)F_k(z) &= cz^{-n_0} \quad \text{(PR property)} \\ \text{and } \frac{1}{M}\sum_{k=0}^{M-1} H_k(zW^g)F_k(z) &= 0, \forall g : 1 \leq g \leq M-1\end{aligned}$$

Now if we replace the filters by $H_k(z^2)$ and $F_k(z^2)$

$$\frac{1}{M}\sum_{k=0}^{M-1} H_k(z^2)F_k(z^2) = cz^{-2n_0} \tag{1}$$

Thus, perfect reconstruction can be obtained if aliasing is cancelled. Let us check if aliasing is cancelled. For $1 \leq g \leq M-1$,

$$\frac{1}{M}\sum_{k=0}^{M-1} H_k(z^2W^{2g})F_k(z^2) = \frac{1}{M}\sum_{k=0}^{M-1} H_k(z^2W^{l})F_k(z^2) \quad \text{where } l = 2g \bmod M$$

$$\frac{1}{M}\sum_{k=0}^{M-1} H_k(z^2W^{2g})F_k(z^2) = \begin{cases} cz^{-2n_0} & 2g \bmod M \\ 0 & else \end{cases}$$

$2g \bmod M = 0 \Longrightarrow M$ is even as $1 \leq g \leq M-1$

Hence, when M is even there is aliasing and perfect reconstruction cannot be obtained. When M is odd, there is no aliasing and hence from (1), perfect reconstruction is obtained