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E9: 252: Mathematical Methods and Techniques in Signal Processing

Instructor: Shayan Srinivasa Garani

Mid Term Exam#1, Fall 2017

Name and SR.No:

Instructions:

- You are allowed only 5 pages of A4 pages written on both sides and a calculator for this exam. No wireless allowed.
- The time duration is 3 hrs.
- There are five main questions. None of them have negative marking.
- Attempt all of them with careful reasoning and justification for partial credit.
- Make any reasonable assumptions if really required.
- Do not panic, do not cheat.
- Good luck!

Question No.	Points scored
1	
2	
3	
4	
5	
Total points	

PROBLEM 1: This problem has 2 parts.

(1) Is the set $1, t, t^2, ..., t^m$ linearly dependent? Justify. (10) pts.

(2) Let $X = L_2[-\pi, \pi]$. Let $S_1 = \text{span}(1, \cos(t), \cos(2t), ...)$ and $S_2 = \text{span}(\sin(t), \sin(2t), ...)$. Examine if $\dim(S_1 \oplus S_2) = \dim(S_1) + \dim(S_2)$. (10 pts.)

1) Consider the set of all polynomials of degree m or less. Let us assume that the set $\{1, t, ..., t^m\}$ is linearly independent. According to our assumption, we get,

$$\alpha_1 + \alpha_2 t + ... + \alpha_{m+1} t^m = 0 \quad -(i)$$

(where $\alpha_1, \alpha_2, ..., \alpha_{m+1}$ are constants)

At least one of α_i for $1 \leq i \leq m+1$ in eq (i) is non-zero and $\alpha_{m+1} \neq 0$. The above equation is true for any value of t.

Hence, the above equation has infinite solutions. But according to fundemental theorem of algebra, the above equation can have exactly 'm' roots which leads to a contradiction. Hence the set $\{1, t, ..., t^m\}$ is linearly independent.

2) The collection $S_a = \{1, \text{Cos}(t), \text{Cos}(2t), \text{Cos}(3t), ...\}$ is orthogonal on $(-\pi, \pi)$

Proof :- $$\int_{-\pi}^{\pi} \text{Cos}(mt) \text{Cos}(nt)\, dt \qquad \text{for } m \neq n$$

$$= \frac{1}{2} \int_{-\pi}^{\pi} \{\text{Cos}((m+n)t) + \text{Cos}((m-n)t)\}\, dt$$

$$= \frac{1}{2} \left[\frac{\text{Sin}((m+n)t)}{m+n} + \frac{\text{Sin}((m-n)t)}{m-n} \right]_{-\pi}^{\pi} = 0$$

Similarly, the collection $S_b = \{\sin(t), \sin(2t), ----\}$ is orthogonal on $[-\pi, \pi]$.

As S_a and S_b are orthogonal sets, they are also linearly independent.

Now $S_1 = \text{Span}(S_a)$ and $S_2 = \text{Span}(S_b)$

Thus S_a and S_b form basis of S_1 and S_2 respectively

It can also be shown that S_1 and S_2 are orthogonal subspaces.

Proof :- $\int_{-\pi}^{\pi} \sin(mt) \cos(nt)\, dt$ $\quad n \geq 0$, $m > 0$ $\quad$ m, n are integers

$$= \frac{1}{2} \int_{-\pi}^{\pi} [\sin(m+n)t + \sin(m-n)t]\, dt$$

$$= -\frac{1}{2} \left[\frac{\cos(m+n)t}{m+n} + \frac{\cos(m-n)t}{m-n} \right]_{-\pi}^{\pi}$$

$$= 0$$

As S_1 and S_2 are orthogonal subspaces, their intersection is 0. $\Rightarrow S_1 \oplus S_2$ is a direct sum

So, $\dim(S_1 \oplus S_2) = \dim(S_1) + \dim(S_2)$

PROBLEM 2: This problem has 2 parts.

(1) Let $e[n]$ denote a white noise sequence, and let $s[n]$ be a sequence uncorrelated with $e[n]$. Examine if $y[n] = s[n]e[n]$ is white. (10 pts.)

(2) Let $x[n]$ be a real stationary white noise sequence with zero mean and variance σ_x^2. Determine the output variance if $x[n]$ is filtered through a cascade of two filters with responses $h_1[n]$ and $h_2[n]$. You can assume that the filters have infinite taps. (10 pts.)

1) $y[n] = s[n]\, e[n]$

Given, $e[n]$ is white

$\Rightarrow E[e[n]] = 0$

$Var[e[n]] = \sigma_x^2$

$E[e[n]\, e^*[n-m]] = 0$

Given, $s[n]$ and $e[n]$ are uncorrelated

$\Rightarrow E[e[n]\, s[n]] = E[e[n]] . E[s[n]]$

Now,

$E[y[n]] = E[s[n]\, e[n]] = E[s[n]]\, E[e[n]] = 0$

(as $E[e[n]] = 0$)

$Var[y[n]] = E[s^2[n]\, y^2[n]]$

$E[y[n]\, y[m]] = E[e[n]\, e[m]\, s[n]\, s[m]]$

$= E[e[n]\, e[m]]\, E[s[n]\, s[m]]$

$= 0$

$Var[y[n]] = E[y[n]^2] = E[s^2[n]\, e^2[n]]$

$= E[s^2[n]]\, E[e^2[n]]$

$= \sigma_s^2\, \sigma_x^2$

$\Rightarrow y[n]$ is white

2) $x[n] \rightarrow$ real stationary white noise sequence with zero mean and variance σ_x^2

$y[n] = h_1[n] * h_2[n] * x[n]$

let $h[n] = h_1[n] * h_2[n]$, then,

$$y[n] = \sum_{m=-\infty}^{\infty} h(m)\, x(n-m)$$

$$E[y[n]] = E\left[\sum_{m=-\infty}^{\infty} h(m)\, x[n-m]\right] = \sum_{m=-\infty}^{\infty} h(m)\, E[x[n-m]] = $$

$$E[y[n]^2] = E\left[\sum_{k=-\infty}^{\infty} h(k)\, x[n-k] \sum_{l=-\infty}^{\infty} h[l]\, x[n-l]\right]$$

$$= \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} h[k]\, h[l]\, E[x[n-k]\, x[n-l]]$$

Now, we know that,

$$E[x[n-k]\, x[n-l]] = \sigma_x^2\, \delta_{k,l}$$

$$\Rightarrow E[y[n]^2] = \left(\sum_{k=-\infty}^{\infty} h[k]^2\right) \sigma_x^2$$

PROBLEM 3: Derive a general form of state space representation for N cascaded LTI systems. Assume that each system in the cascade has a state space representation $\mathbf{A}_i, \mathbf{b}_i, \mathbf{c}_i^T, d_i = 0$ for $0 \leq i \leq N-1$. (10 pts.)

consider the systems in cascade as follows:

$$x_0(n) \to \boxed{(A_1, b_1, c_1^T)} \xrightarrow{x_1(n)} \boxed{(A_2, b_2, c_2^T)} \xrightarrow{x_2(n)} \cdots \xrightarrow{x_{N-1}(n)} \boxed{(A_N, b_N, c_N^T)} \to y(n) = x_N(n)$$

Now, we know that

$$w_1(n+1) = A_1 w_1(n) + b_1 x_0(n)$$
$$x_1(n) = c_1^T w_1(n)$$

For $i \in \{2, \dots, N\}$,

$$w_i(n+1) = A_i w_i(n) + b_i x_{i-1}(n)$$
$$x_i(n) = c_i^T w_i(n)$$

$$\Rightarrow \quad w_i(n+1) = A_i w_i(n) + b_i c_{i-1}^T w_{i-1}(n)$$

Let us consider the state vector to be $[w_N(n) \;\; w_{N-1}(n) \; \cdots \; w_2(n) \;\; w_1(n)]$

then,

$$\begin{bmatrix} w_N(n+1) \\ w_{N-1}(n+1) \\ \vdots \\ w_1(n+1) \end{bmatrix} = \begin{bmatrix} A_N & b_N c_{N-1}^T & & & \\ & A_{N-1} & b_{N-1} c_{N-2}^T & & \\ & & \ddots & \ddots & \\ & & & \ddots & b_2 c_1^T \\ & & & & A_1 \end{bmatrix} \begin{bmatrix} w_N(n) \\ w_{N-1}(n) \\ \vdots \\ w_2(n) \\ w_1(n) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ b_1 \end{bmatrix} x_0(n)$$

and

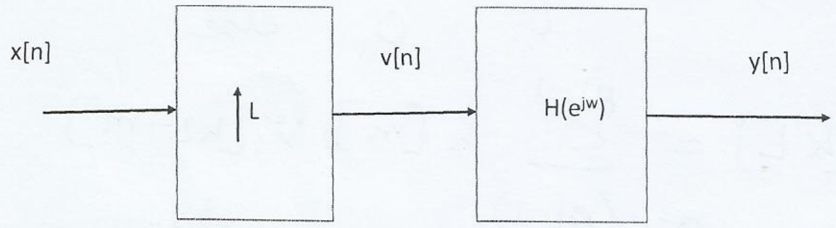
$$y(n) = x_N(n) = c_N^T w_N(n)$$
$$= [c_N^T \;\; 0 \; \cdots \; 0] \begin{bmatrix} w_N(n) \\ w_{N-1}(n) \\ \vdots \\ w_1(n) \end{bmatrix}$$

$\Rightarrow$ for the cascaded system,

$$A = \begin{bmatrix} A_N & b_N c_{N-1}^T & & & \\ & A_{N-1} & b_{N-1} c_{N-2}^T & & \\ & & \ddots & \ddots & \\ & & & \ddots & b_2 c_1^T \\ & & & & A_1 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ b_1 \end{bmatrix} \quad \text{and } c = \begin{bmatrix} c_N \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

and $d = 0$.

PROBLEM 4: The system shown in Figure approximately interpolates a discrete time sequence $x[n]$ by a factor L. Suppose that the linear filter has impulse response $h[n] = h[-n]$ and $h[n] = 0$ for $|n| > (RL-1)$, where R and L are integers; i.e., the impulse response is symmetric and of length $2RL - 1$ samples.



(1) How much delay must be inserted to make the system causal? (5 pts.)

(2) What conditions must be satisfied by $h[n]$ such that $y[n] = x[\frac{n}{L}]$ for $n = 0, \pm L, \pm 2L, ...$? (5 pts.)

(3) By exploiting the symmetry of the impulse response of the filter, show that each sample of $y[n]$ can be computed with no more than RL multiplications. (5 pts.)

(4) By taking advantage of the fact that multiplications by zero need not be done, show that only $2R$ multiplications per output sample are required. (5 pts.)

$$V(n) = \begin{cases} x(n/L) & n \bmod L = 0 \\ 0 & \text{else} \end{cases}$$

$$V(z) = X(z^L)$$

$$Y(z) = X(z^L)\,H(z)$$

1) $$y[n] = V[n] * h[n] = \sum_{m=-\infty}^{\infty} h[m]\, V[n-m] = \sum_{m=-(RL-1)}^{RL-1} h[m]\, V[n-m]$$

$n-m \bmod L = 0$

Can $n+RL-1 \bmod L = 0$?

Yes, when $n-1 = RL$

$\therefore$ As we have term upto $x[n+RL-1] \Rightarrow$ delay of $(RL-1)$ is needed at $V[n]$

$\Rightarrow$ Does $n+RL-1 \bmod L = 0$

If so, at $x[n]$, the delay must be $\frac{n+RL-1}{L}$

$\Rightarrow$ delay of R is needed.

4.2) $$y[n] = \sum_{m=-(RL-1)}^{RL-1} h[m]\, v[n-m]$$

Now, $v[n] = \begin{cases} x[n/L] & n \bmod L = 0 \\ 0 & \text{else} \end{cases}$

$$v[n-m] = \begin{cases} x\left[\frac{n-m}{L}\right], & n-m \bmod L = 0 \\ 0 & \text{else} \end{cases}$$

$$y[kL] = \sum_{m=-(RL-1)}^{RL-1} h[m]\, v[kL-m]$$

$$v(kL-m] = \begin{cases} x\left[\frac{kL-m}{L}\right] & kL-m \bmod L = 0 \\ 0 & \text{else} \end{cases}$$

$$= \begin{cases} x[k-m/L] & m \bmod L = 0 \\ 0 & \text{else} \end{cases}$$

$$\Rightarrow y[kL] = \sum_{m=-(R-1}^{R-1} h(mL)\, x[k-m]$$

4.3) $$y[n] = \sum_{m=-(RL-1)}^{RL-1} h[m]\, v[n-m]$$

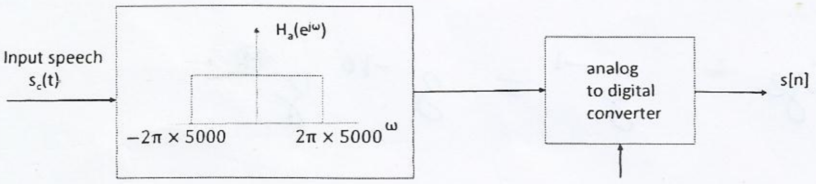
$$= \sum_{m=-(RL-1)}^{-1} h[-m]\, v[n-m] + \sum_{m=0}^{RL-1} h[m]\, v[n-m]$$

$$= h[0]\, v[n] + \sum_{m=1}^{RL-1} h[m]\left[v[n-m] + v[n+m]\right]$$

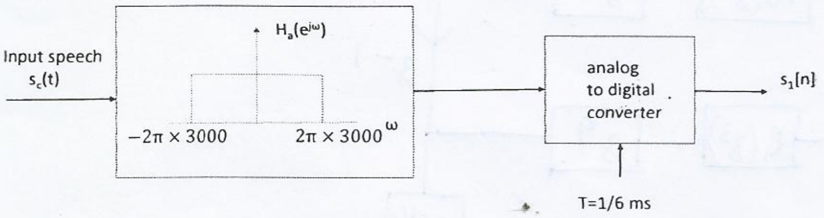
$\therefore$ Only RL multiplications are required

4.4) From $-(RL-1)$ to $(RL-1)$, there are 2R multiples of L and hence only 2R multiplications per output sample are required.

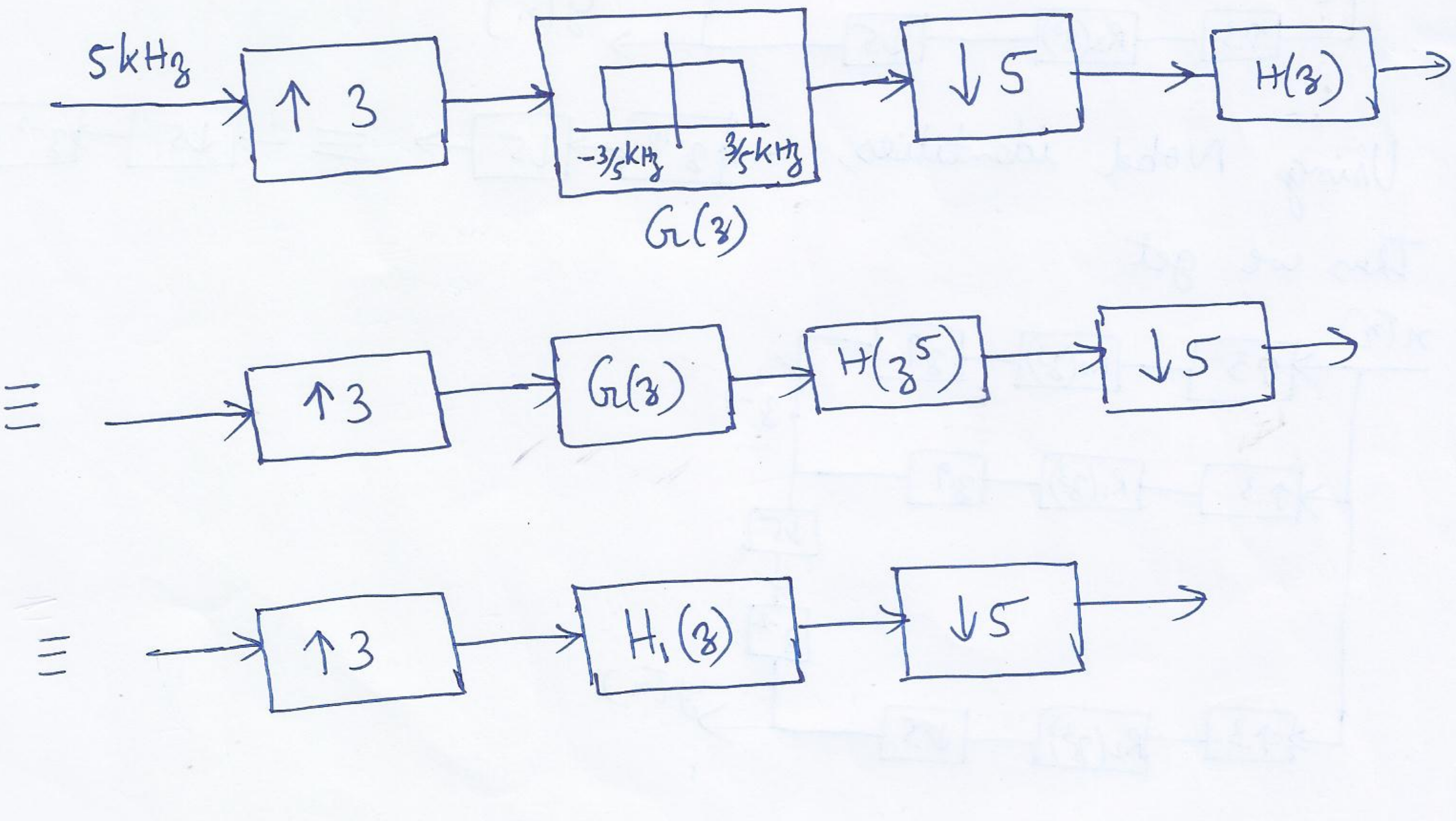
PROBLEM 5: Suppose you obtained a sequence $s[n]$ by filtering a speech signal $s_c(t)$ with a continuous time low pass filter with a cutoff of 5 KHz and then sampling it at 10 KHz rate shown in Figure (a). Unfortunately, the speech signal $s_c(t)$ is destroyed once $s[n]$ was stored on a disk drive. Later you decided that you should have followed the process in Figure (b). Develop a method to obtain $s_1[n]$ from $s[n]$ using appropriate processing. Suppose it was required to filter $s_1[n]$ through a discrete time filter $H(z)$ for any post processing. Show how you will realize this efficiently using signals $s[n]$ and $H(z)$. (30 pts.)



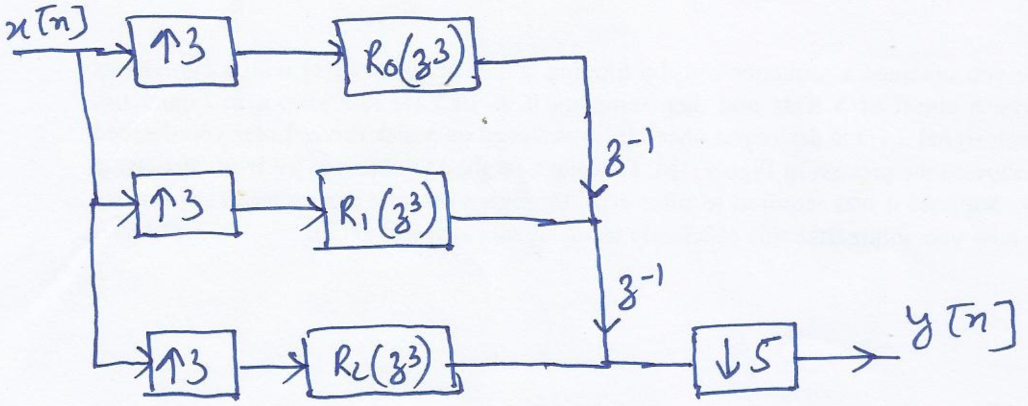
(a)



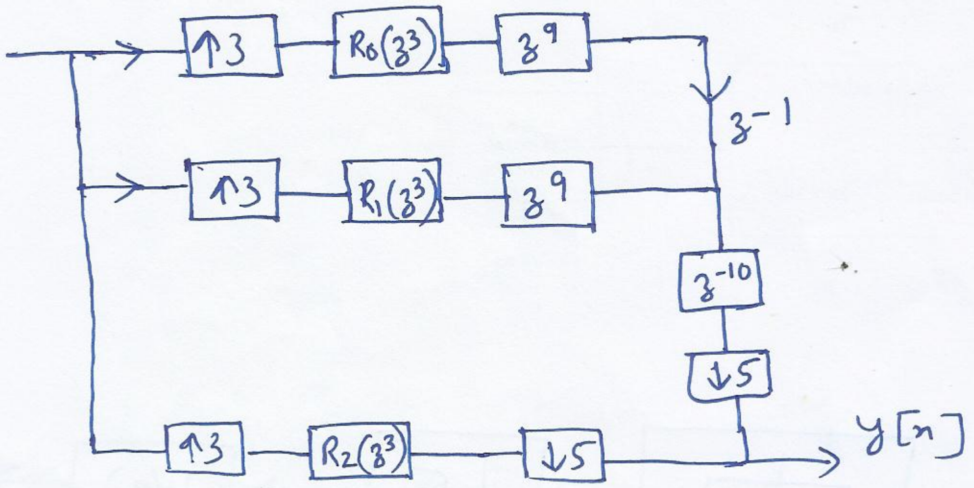
(b)



We need polyphase decomposition similar to Hsiao's work.

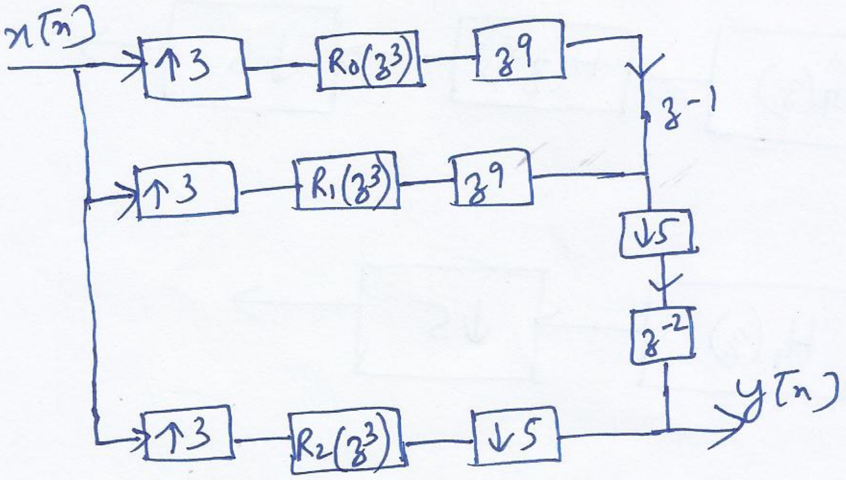


Realize $z^{-1} = z^{-10} z^{9}$

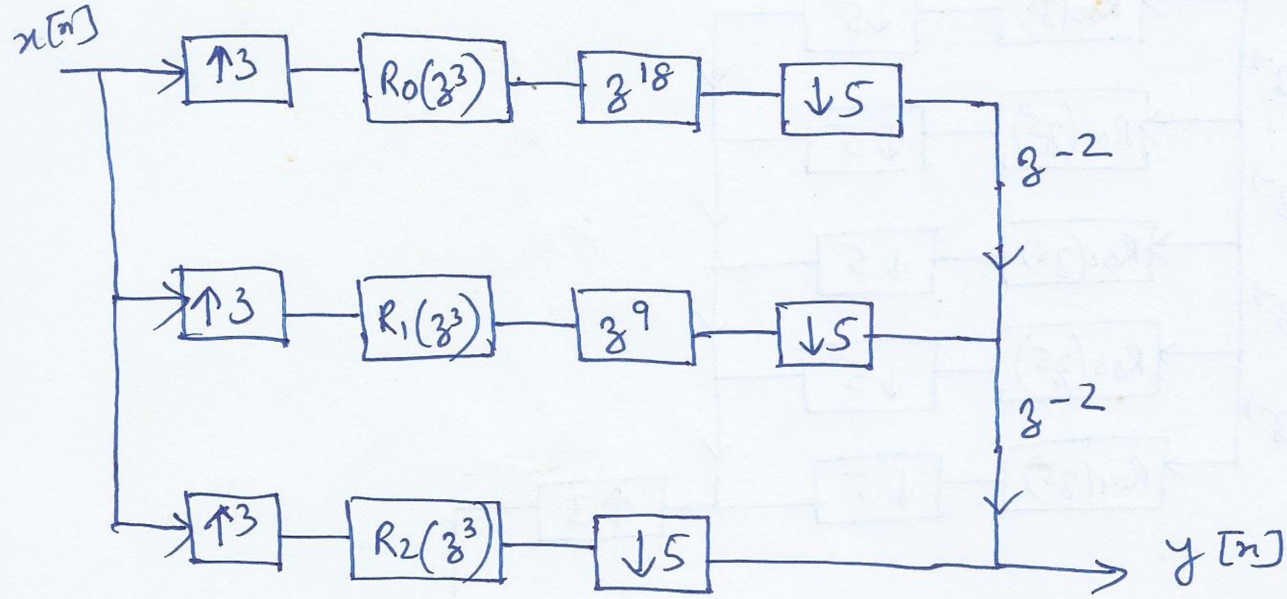


Using Noble identities, $\rightarrow \boxed{z^{-10}} \rightarrow \boxed{\downarrow 5} \rightarrow \;\equiv\; \rightarrow \boxed{\downarrow 5} \rightarrow \boxed{z^{-2}} \rightarrow$

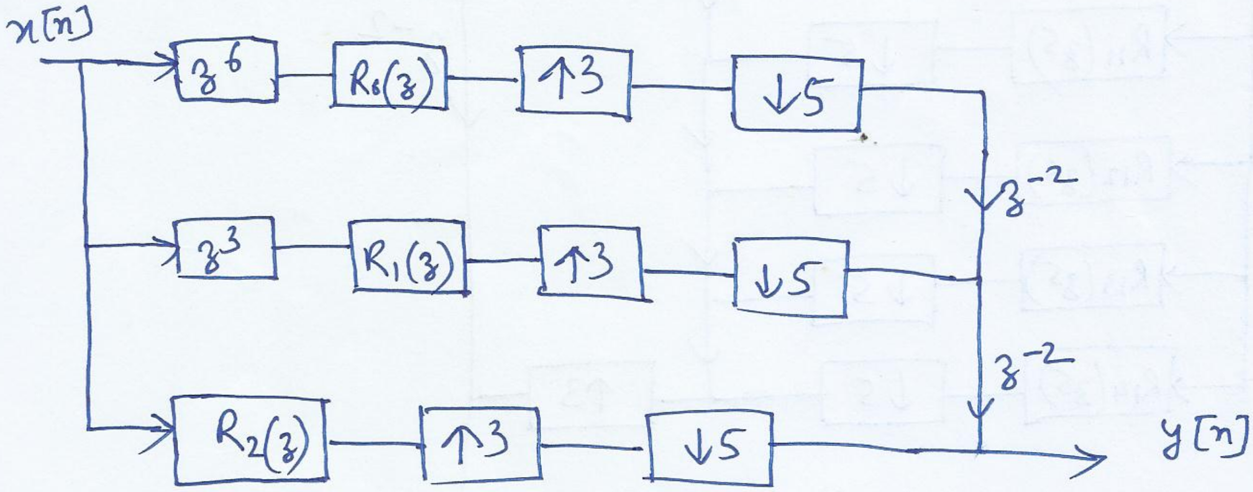
Thus we get



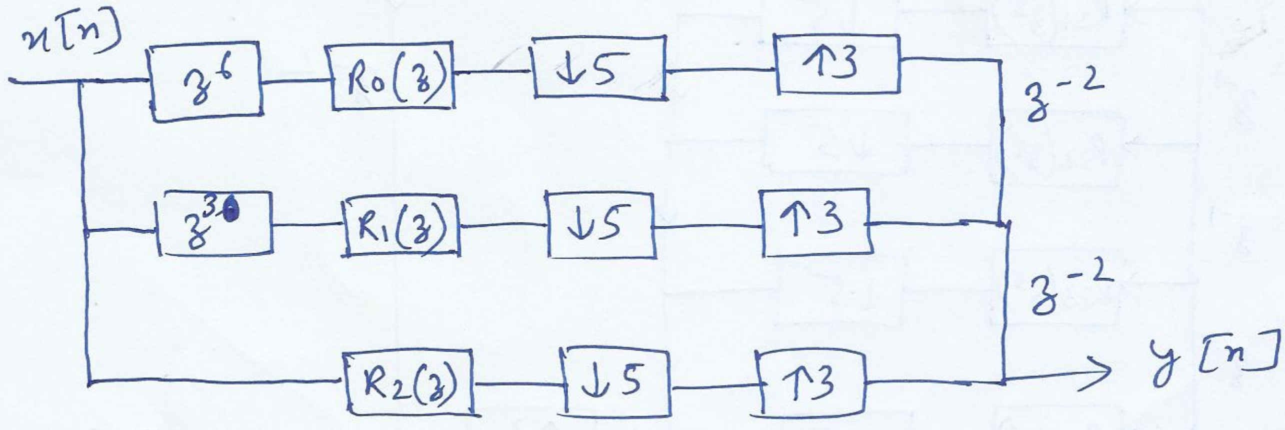
Proceeding similarly as above our architecture ~~is~~ becomes



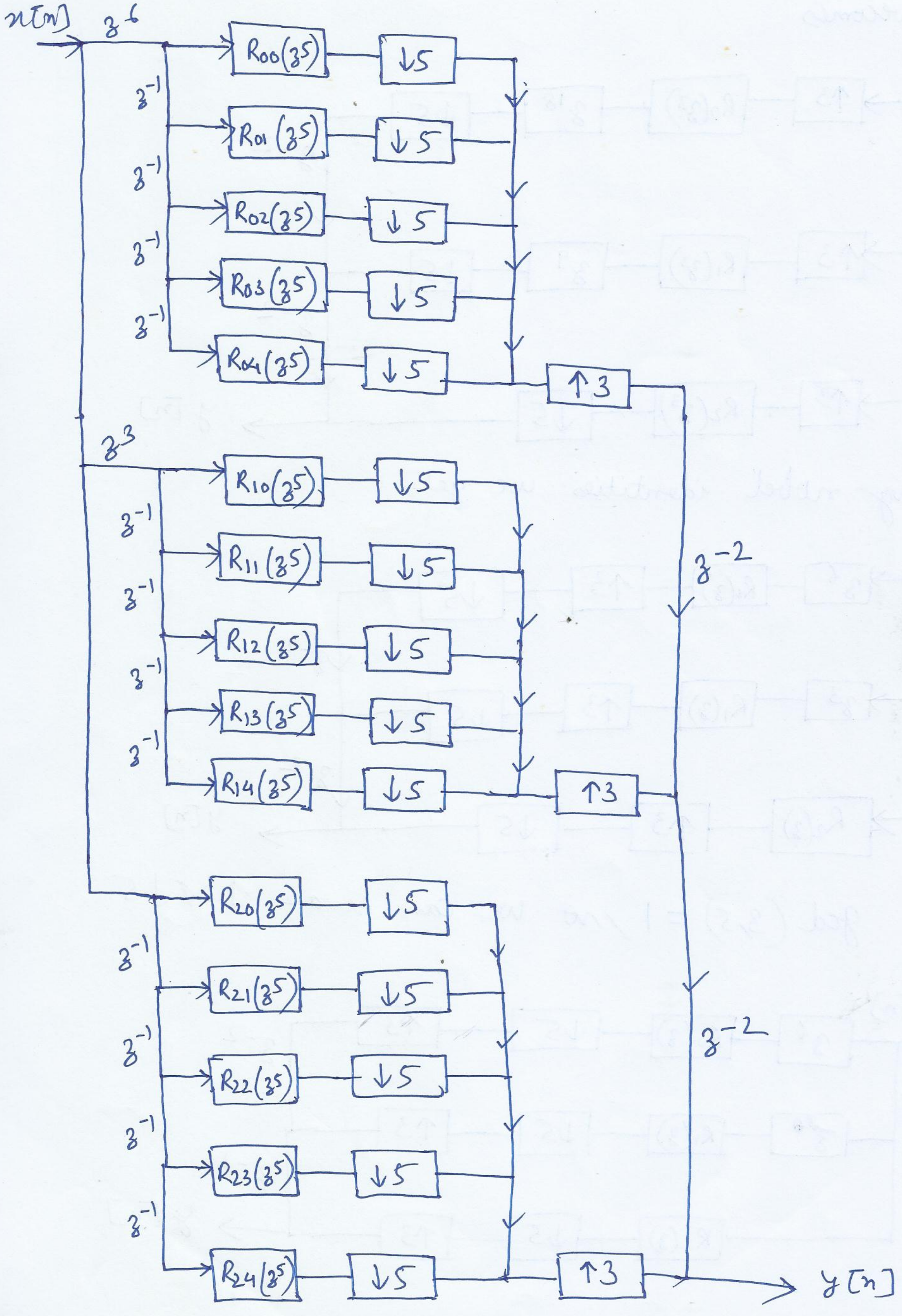
Using nobel identities we get,



gcd (3,5) = 1, so we can swap ↑3 & ↓5



Using polyphase decomposition again we get,



Final architecture