E9-252: Mathematical Methods and Techniques in Signal Processing
Homework 3 Solutions

Instructor: Shayan Garani Srinivasa
Solutions Prepared By: Chaitanya Kumar Matcha

23 November 2016

Problem 4.17
Part a:

$$x_1[2n] = x[n]$$
$$x_1[2n+1] = 0$$

$$y^{(1)}[n] = \sum_{k=-\infty}^{\infty} x_1[k]\, g[n-k]$$
$$= \sum_{l=-\infty}^{\infty} x_1[2l]\, g[n-2l] \quad (\because x_1[2n+1] = 0)$$
$$y^{(1)}[n] = \sum_{l=-\infty}^{\infty} x[l]\, g[n-2l]$$

We want $y^{(1)}[2n] = x[2n]\, \forall n$

$$\implies x[2n] = \sum_{l=-\infty}^{\infty} x[l]\, g[2n-2l]$$
$$\implies x[2n](1-g[0]) - \sum_{l\neq 0} x[l]\, g[2n-2l] = 0$$

Since this is true for all $x[n]$, we have

$$g[0] = 1,$$
$$g[2n] = 0, \quad n \in \mathbb{Z} \setminus \{0\}.$$

Part b:
We have

$$y^{(1)}[2n] = x[n]$$
$$\implies y^{(2)}[4n] = y^{(1)}[2n] = x[n].$$

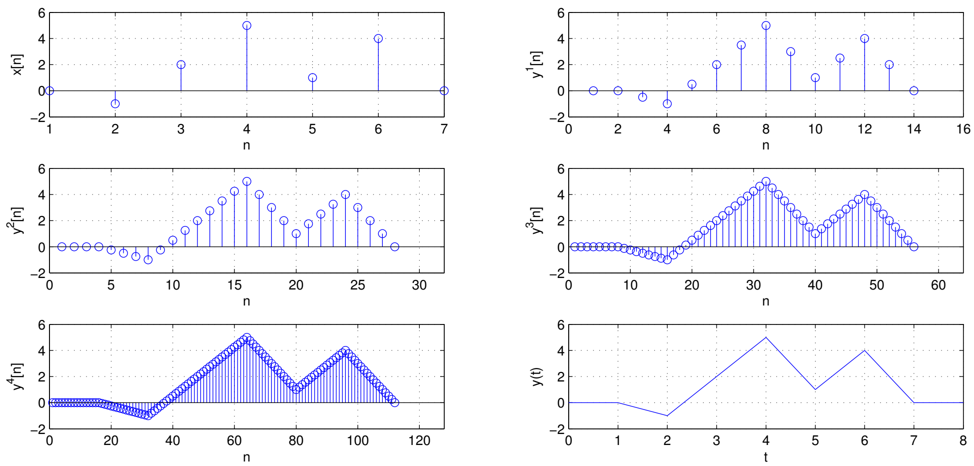
Assume $y^{(k)}\left[2^k n\right] = x[n]$. This implies $y^{(k+1)}\left[2^{k+1} n\right] = y^{(k)}\left[2^k n\right] = x[n]$. Therefore, by induction,

$$y^{(k)}\left[2^k n\right] = x[n] \qquad \forall k = 0, 1, \cdots; \quad n \in \mathbb{Z}.$$

Part c:

$$G(z) = \frac{1}{2}z + 1 + \frac{1}{2}z^{-1}.$$

Following shows $y^{(k)}[n]$ for an example $x[n]$ and different choices of k.



Part d:

As $k \to \infty$, the function $y^{(\infty)}(t)$ is a continuous function that linearly interpolates the samples $x[n]$ and $x[n+1]$ in the interval $t \in [n, n+1], \quad n \in \mathbb{Z}$.

From Problem 4.18, the closed form expression for $y^{(\infty)}(t)$ is

$$y^{(\infty)}(t) = \sum_{n=-\infty}^{\infty} x[n]\, g^{(\infty)}(t-n)$$

where

$$g^{(\infty)}(t) = \begin{cases} 1-x, & 0 \le x \le 1 \\ 1+x, & -1 \le x < 0 \\ 0 & \text{otherwise.} \end{cases}$$

From the figure we can see that $y^{(\infty)}(t)$ is continuous everywhere but need not be differentiable everywhere.

Problem 4.18

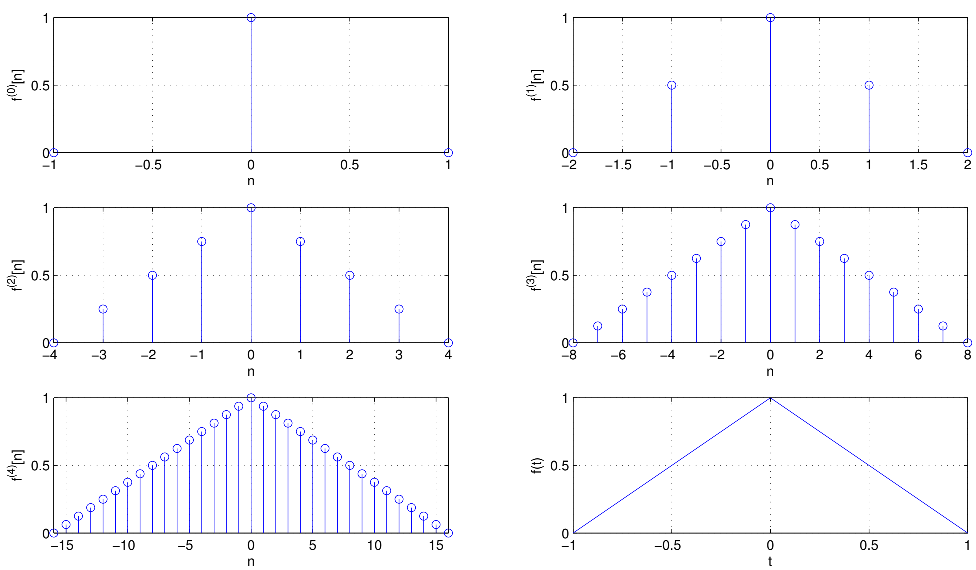
Part a:

We have

$$f^{(0)}[n] = \delta[n]$$
$$f^{(1)}[n] = c[n] * f^{(0)}[2n] = c[n].$$
$$f^{(k)}[n] = c[n] * f^{(k-1)}[2n].$$

where

$$C(z) = \frac{1}{2}z + 1 + \frac{1}{2}z^{-1}.$$



From the plots, we can observe that

$$f^{(\infty)}(t) = \begin{cases} 1-x, & 0 \le x \le 1 \\ 1+x, & -1 \le x < 0 \\ 0, & \text{otherwise.} \end{cases}$$

You can prove this formally as follows:

$c[n]$ and $f^{(0)}[n]$ are even functions. Therefore, $f^{(k)}[n]$ and $f^{(\infty)}(t)$ are all even functions. We will prove only for $t \ge 0$.

Observation 1: $f^{(k)}[n]$ is non-zero only for $-\left(2^k - 1\right) \le n \le \left(2^k - 1\right)$ and $f^{(k)}[n] \ge 0$.

Proof: (By induction) This is true for $k = 0, 1$. Assume that it is true for $f^{(k)}[n]$ i.e., $F^{(k)}(z)$ is a polynomial with $-\left(2^k - 1\right)$ and $\left(2^k - 1\right)$ as the least and highest order of z and all coefficients positive. We have

$$F^{(k+1)}(z) = C(z)\, F^{(k)}\left(z^2\right)$$

$F^{(k)}\left(z^2\right)$ has $-\left(2^{k+1} - 2\right)$ and $\left(2^{k+1} - 2\right)$ as the least and highest order of z. Since $C(z) = \frac{1}{2}z + 1 + \frac{1}{2}z^{-1}$ and $F^{(k)}\left(z^2\right)$ has all positive coefficients, therefore $F^{(k+1)}(z)$ has $-\left(2^{k+1} - 1\right)$ and $\left(2^{k+1} - 1\right)$ as the least and highest order of z.

Observation 2: $f^{(k)}[n] = 1 - \frac{n}{2^k}$, $0 \le n \le \left(2^k - 1\right)$

Proof: (By induction) This is true for $k = 0, 1$. Assume that it is true for $f^{(k)}[n]$.

$$
\begin{aligned}
f^{(k+1)}[n] &= \sum_{i=-\infty}^{\infty} f^{(k)}[i]\, c[n-2i] \\
&= \begin{cases} f^{(k)}\left[\frac{n}{2}\right] & , n \text{ is even} \\ \frac{1}{2} f^{(k)}\left[\frac{n-1}{2}\right] + \frac{1}{2} f^{(k)}\left[\frac{n+1}{2}\right] & , n \text{ is odd.} \end{cases} \\
&= \begin{cases} 1 - \frac{n}{2^{k+1}} & , n \text{ is even} \\ \frac{1}{2}\left(1 - \frac{n-1}{2^k}\right) + \frac{1}{2}\left(1 - \frac{n+1}{2^k}\right) & , n \text{ is odd.} \end{cases} \\
&= \begin{cases} 1 - \frac{n}{2^{k+1}} & , n \text{ is even} \\ 1 - \frac{n}{2^{k+1}} & , n \text{ is odd.} \end{cases}
\end{aligned}
$$

$$
\begin{aligned}
f^{(\infty)}\left(\frac{n}{2^k}\right) &= f^{(k)}(n) = 1 - \frac{n}{2^k} \\
\Longrightarrow\ f^{(\infty)}(t) &= 1 - t, \quad t = 0, \frac{1}{2^k}, \frac{2}{2^k}, \frac{3}{2^k} \cdots; \quad \forall k.
\end{aligned}
$$

Taking $k \to \infty$, we have $f^{(\infty)}(t) = 1 - t, \quad t \in [0,1]$. Since $f^{(\infty)}(t)$ is an even function, we have

$$
f^{(\infty)}(t) = \begin{cases} 1 - x, & 0 \le x \le 1 \\ 1 + x, & -1 \le x < 0 \\ 0, & \text{otherwise.} \end{cases}
$$

Part b:

$$
\begin{aligned}
g^{(0)}[n] &= s[n] \\
g^{(k)}\left[\frac{2n}{2^k}\right] &= g^{(k-1)}\left[\frac{n}{2^{k-1}}\right], \\
g^{(k)}\left[\frac{2n+1}{2^k}\right] &= \frac{1}{2} g^{(k-1)}\left[\frac{n}{2^{k-1}}\right] + \frac{1}{2} g^{(k-1)}\left[\frac{n+1}{2^{k-1}}\right].
\end{aligned}
$$

Notice that $g^{(\infty)}(t)$ is linear in $s[n]$ i.e., if $g_1(t)$ is obtained with $g^{(0)}[n] = s_1[n]$ and $g_2^{(\infty)}(t)$ is obtained with $g^{(0)}[n] = s_2[n]$, then with $g^{(0)}[n] = as_1[n] + bs_2[n]$ results in $g^{(\infty)}(t) = ag_1^{(\infty)}(t) + bg_2^{(\infty)}(t)$ for any constants a, b.

Also, if $g(t)$ is obtained with $g^{(0)}[n] = s[n]$, then $g^{(0)}[n] = s[n]$ will result in $g^{(\infty)}(t) = g(t-k)$.

From the part 1, we know that $g^{(0)}[n] = \delta[n]$ results in $g(t) = f^{(\infty)}(t)$. Therefore,

$$
\begin{aligned}
g^{(0)}[n] = s[n] &= \sum_{i=-\infty}^{\infty} \delta[i-n]\, g[i] \\
\Longrightarrow\ g^{(\infty)}(t) &= \sum_{i=-\infty}^{\infty} g[i]\, f^{(\infty)}(i-t).
\end{aligned}
$$

Part c:

We have

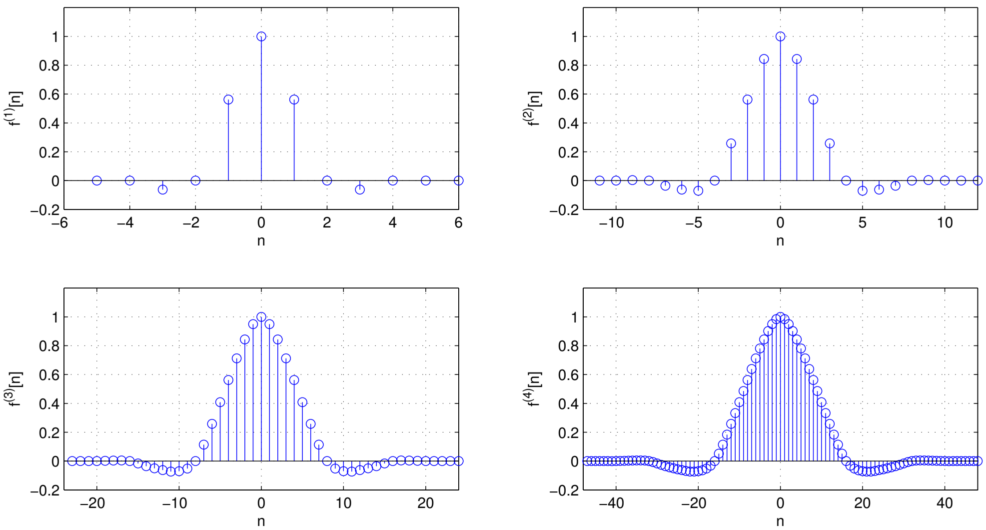
$$
f(t) = f(2x) + \frac{9}{16}\left[f(2x+1) + f(2x-1)\right] - \frac{1}{16}\left[f(2x+3) + f(2x-3)\right].
$$

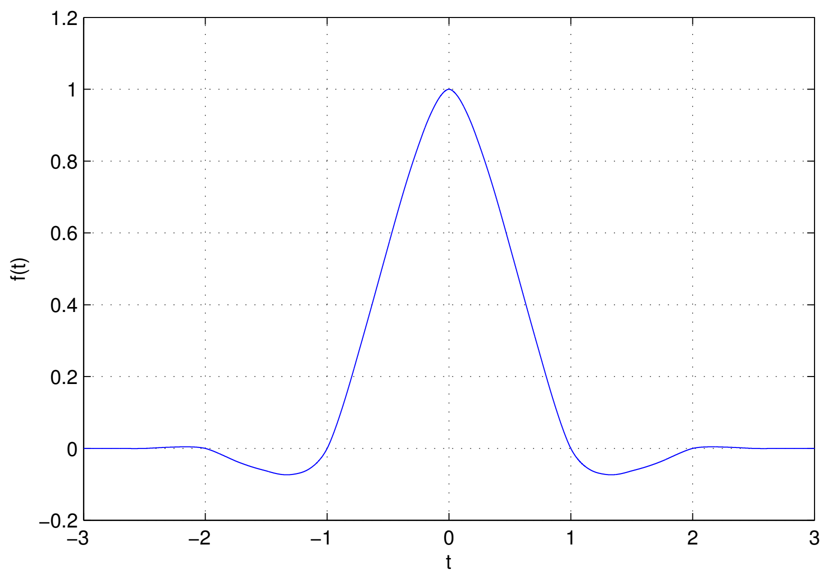
i.e.,

$$
f^{(k)}[n] = c[n] * f^{(k-1)}[2n]
$$

where $C(z) = 1 + \frac{9}{16}\left[z + z^{-1}\right] - \frac{1}{16}\left[z^3 + z^{-3}\right]$.

Following shows $f^{(k)}[n]$ for different values of k as well as $f^{(\infty)}(t)$.

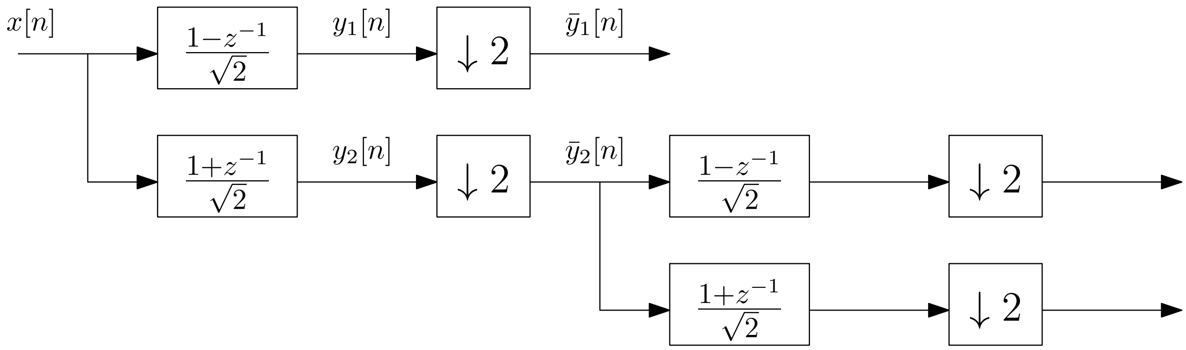




Similar to part b, we have

$$g^{(\infty)}(t) = \sum_{i=-\infty}^{\infty} g[i] \, f^{(\infty)}(i-t).$$

Problem 2



Let L be the length of the input sequence i.e., $x[n]$ is restricted to $0 \le n \le L-1$. Therefore,

$$\begin{aligned} y_1[0] &= \frac{1}{\sqrt{2}}\left(x[0] - x[L-1]\right) \\ y_1[n] &= \frac{1}{\sqrt{2}}\left(x[n] - x[n-1]\right), \quad n = 1, 2, \cdots, L-1. \end{aligned}$$

$$\begin{aligned} y_2[0] &= \frac{1}{\sqrt{2}}\left(x[0] + x[L-1]\right) \\ y_2[n] &= \frac{1}{\sqrt{2}}\left(x[n] + x[n-1]\right), \quad n = 1, 2, \cdots, L-1. \end{aligned}$$

Energy at the input is

$$E_{ip} = \sum_{n=0}^{L-1} \left(x[n]\right)^2$$

<u>Case L is odd:</u>

$$\bar{y}_1[n] = \begin{cases} \frac{1}{\sqrt{2}}\left(x[0] - x[L-1]\right), & n = 0 \\ \frac{1}{\sqrt{2}}\left(x[2n] - x[2n-1]\right), & n = 1, \cdots, \frac{L-1}{2} \end{cases}$$

$$\bar{y}_2[n] = \begin{cases} \frac{1}{\sqrt{2}}\left(x[0] + x[L-1]\right), & n = 0 \\ \frac{1}{\sqrt{2}}\left(x[2n] + x[2n-1]\right), & n = 1, \cdots, \frac{L-1}{2} \end{cases}$$

$$\Longrightarrow \left(\bar{y}_1[n]\right)^2 + \left(\bar{y}_2[n]\right)^2 = \begin{cases} \left(x[0]\right)^2 + \left(x[L-1]\right)^2, & n = 0 \\ \left(x[2n]\right)^2 + \left(x[2n-1]\right)^2, & n = 1, \cdots, \frac{L-1}{2} \end{cases}$$

Energy at the output is

$$\begin{aligned} \sum_{n=0}^{\frac{L-1}{2}} \left(\left(\bar{y}_1[n]\right)^2 + \left(\bar{y}_2[n]\right)^2\right) &= \left(x[0]\right)^2 + \left(x[L-1]\right)^2 \\ &\quad + \left(x[2]\right)^2 + \left(x[1]\right)^2 + \left(x[4]\right)^2 + \left(x[3]\right)^2 + \cdots + \left(x[L-1]\right)^2 + \left(x[L-2]\right)^2 \\ &= \left(x[L-1]\right)^2 + \sum_{n=0}^{L-1} \left(x[n]\right)^2 \\ &> E_{ip}. \end{aligned}$$

Therefore the energy is not conserved.
<u>Case L is even:</u>

$$\bar{y}_1[n] = \begin{cases} \frac{1}{\sqrt{2}}\left(x[0] - x[L-1]\right), & n = 0 \\ \frac{1}{\sqrt{2}}\left(x[2n] - x[2n-1]\right), & n = 1, \cdots, \frac{L-2}{2} \end{cases}$$

$$\bar{y}_2[n] = \begin{cases} \frac{1}{\sqrt{2}}\left(x[0] + x[L-1]\right), & n = 0 \\ \frac{1}{\sqrt{2}}\left(x[2n] + x[2n-1]\right), & n = 1, \cdots, \frac{L-2}{2} \end{cases}$$

$$\implies (\bar{y}_1[n])^2 + (\bar{y}_2[n])^2 = \begin{cases} (x[0])^2 + (x[L-1])^2, & n = 0 \\ (x[2n])^2 + (x[2n-1])^2, & n = 1, \cdots, \frac{L-2}{2} \end{cases}$$

Energy at the output is

$$\begin{aligned} \sum_{n=0}^{\frac{L-1}{2}} \left((\bar{y}_1[n])^2 + (\bar{y}_2[n])^2\right) &= (x[0])^2 + (x[L-1])^2 \\ &\quad + (x[2])^2 + (x[1])^2 + (x[4])^2 + (x[3])^2 + \cdots + (x[L-2])^2 + (x[L-3])^2 \\ &= \sum_{n=0}^{L-1} (x[n])^2 \\ &= E_{ip}. \end{aligned}$$

Therefore the energy is conserved.

When L is even, the input to the second stage is of length $\frac{L}{2}$. For the energy conservation in the second stage, $\frac{L}{2}$ must be even, i.e., L is a multiple of 2^2.

Extending it to k-stage wavelet filter bank, energy is conserved if L is a multiple of 2^k. If the input length is not multiple of 2^k, zeros can be padded to make the input length multiple of 2^k.