E9-252: Mathematical Methods and Techniques in Signal Processing
Homework 1 Solutions

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Problem 1 (Moon and Stirling, 1.4.15)

$$H(z) = \sum_{k=1}^{p} \frac{N_k}{z - p_k}$$

a) Draw a block diagram representing the partial fraction expansion, by using the fact that,

$$\frac{Y(z)}{F(z)} = \frac{1}{z - p}$$

b) Let x_i, $i = 1,\ 2\ , ...,\ p$ denote the outputs of the delay elements. Show that the system can be intro state-space form with

$$A = \begin{bmatrix} p_1 & 0 & 0 & ... & 0 \\ 0 & p_2 & 0 & ... & 0 \\ \vdots & \vdots & \vdots & \vdots & 0 \\ 0 & 0 & 0 & ... & p_p \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \quad \mathbf{c} = \begin{bmatrix} N_1 \\ N_2 \\ \vdots \\ N_p \end{bmatrix} \quad d = b_0$$

A matrix A in this from is said to be a diagonal matrix.

c) Determine the partial fraction expansion of

$$H(z) = \frac{1 - 2z^{-1}}{1 + 0.5z^{-1} + 0.06z^{-2}}$$

and draw the block diagram based upon it. Determine (A, **b**, **c**, d).

d) When there are repeated roots, things are slightly more complicated. Consider for simplicity, a root appearing only twice. Determine the partial fraction expansion of

$$H(z) = \frac{1 + z^{-1}}{(1 - 0.2z^{-1})(1 - 0.5z^{-1})^2}$$

e) Draw the block diagram corresponding to $H(z)$ in parital fraction form using only three delay elements.

f) Show that the state variables can be chosen so that

b

$$A = \begin{bmatrix} 0.5 & 0 & 0 \\ 1 & 0.5 & 0 \\ 0 & 0 & 0.2 \end{bmatrix}$$

Solution:

a)

$$\frac{Y(z)}{F(z)} = \frac{1}{z-p}$$

$$\begin{aligned} H(z) &= \sum_{k=1}^{p} \frac{N_k}{z-p_k} \\ &= \frac{N_1 z^{-1}}{1-p_1 z^{-1}} + \frac{N_2 z^{-1}}{1-p_2 z^{-1}} + \ldots + \frac{N_p z^{-1}}{1-p_p z^{-1}} \end{aligned} \tag{1}$$

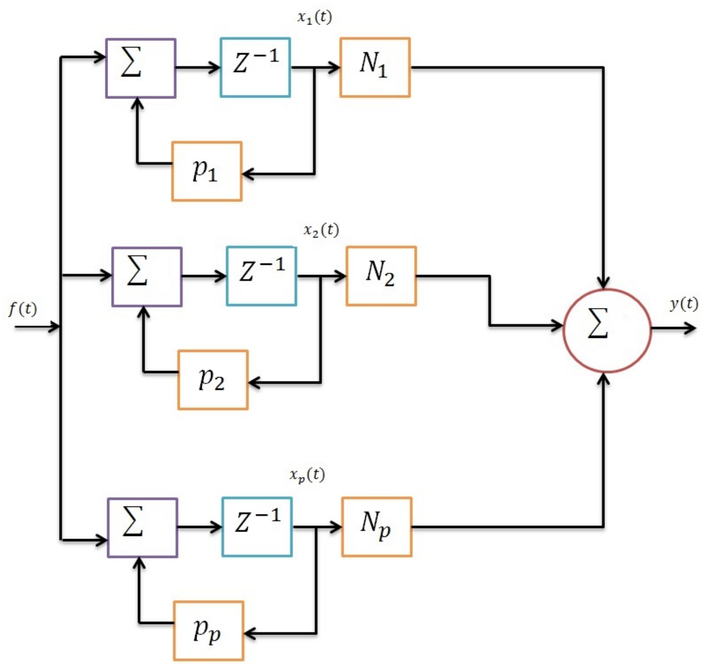


Figure 1

b)

From the block diagram in the last question we have,

$$x_i(t) = f(t-1) + p_i x_i(t-1)$$

Which also means,

$$x_i(t+1) = f(t) + p_i x_i(t) \tag{2}$$

and,

$$y(t) = N_1 x_1(t) + N_2 x_2(t) + \ldots + N_p x_p(t). \tag{3}$$

In matrix notations we have,

$$\begin{bmatrix} X_1(t+1) \\ X_2(t+1) \\ \vdots \\ X_p(t+1) \end{bmatrix} = \begin{bmatrix} p_1 & 0 & 0 & ... & 0 \\ 0 & p_2 & 0 & ... & 0 \\ \vdots & \vdots & \vdots & \vdots & 0 \\ 0 & 0 & 0 & ... & p_p \end{bmatrix} \begin{bmatrix} X_1(t) \\ X_2(t) \\ \vdots \\ X_p(t) \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} f(t) \tag{4}$$

$$y(t) = \begin{bmatrix} N_1 & N_2 & \cdots & N_p \end{bmatrix} \begin{bmatrix} X_1(t) \\ X_2(t) \\ \vdots \\ X_p(t) \end{bmatrix} f(t) \tag{5}$$

This gives us,

$$A = \begin{bmatrix} p_1 & 0 & 0 & ... & 0 \\ 0 & p_2 & 0 & ... & 0 \\ \vdots & \vdots & \vdots & \vdots & 0 \\ 0 & 0 & 0 & ... & p_p \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \quad \mathbf{c} = \begin{bmatrix} N_1 \\ N_2 \\ \vdots \\ N_p \end{bmatrix} \quad d = 0.$$

c)

$$\begin{aligned} H(z) &= \frac{1 - 2z^{-1}}{1 + 0.5z^{-1} + 0.06z^{-2}} \\ &= \frac{z^2 - 2z}{z^2 + 0.5z + 0.06} \\ &= \frac{z(z-2)}{(z+0.2)(z+0.3)} \end{aligned} \tag{6}$$

Thus,

$$H(z) = \frac{z(z-2)}{(z+0.2)(z+0.3)} = 1 + \frac{A}{z+0.2} + \frac{B}{z+0.3} \tag{7}$$

Solving, we get

$$A = 4.4 \text{ and } B = -6.9$$

which gives us,

$$H(z) = 1 + \frac{4.4}{z - (-0.2)} - \frac{6.9}{z - (-0.3)} \tag{8}$$

which finally gives us the following

$$A = \begin{bmatrix} -0.2 & 0 \\ 0 & -0.3 \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \mathbf{c} = \begin{bmatrix} 4.4 \\ -6.9 \end{bmatrix} \quad d = 1. \tag{9}$$

d)

$$H(z) = \frac{1 + z^{-1}}{(1 - 0.2z^{-1})(1 - 0.5z^{-1})^2} = \frac{A}{z - 0.2} + \frac{B}{z - 0.5} + \frac{C}{(z-0.5)^2} + 1$$

solving for A, B, and C

$$H(z) = \frac{0.533}{z - 0.2} + \frac{1.667}{z - 0.5} + \frac{1.25}{(z-0.5)^2} + 1 \tag{10}$$

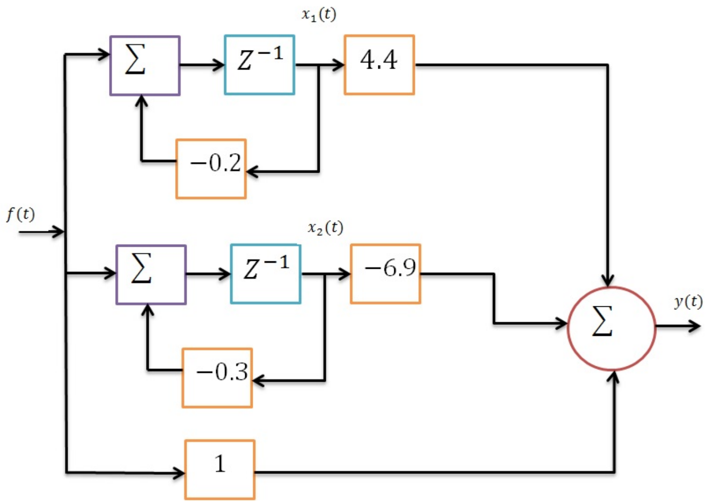


Figure 2

e)

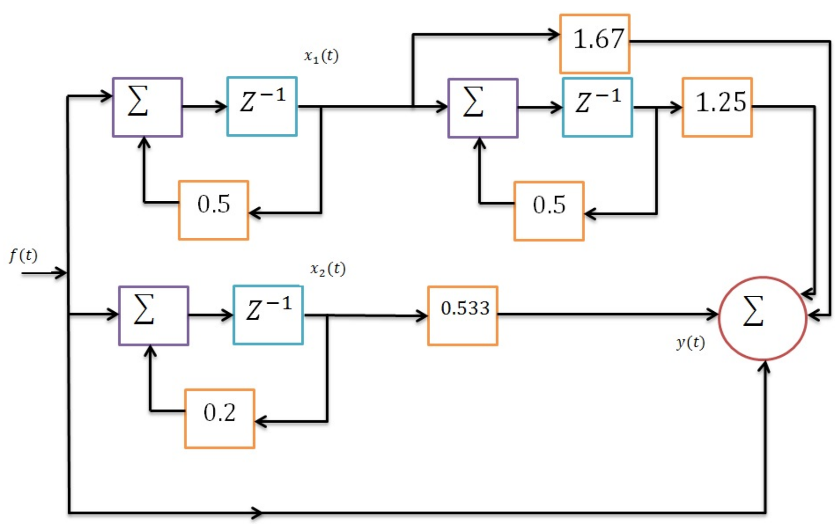


Figure 3

f)

From the last figure we get the following equations

$$\begin{aligned} x_1(t+1) &= 0.5x_1(t) + f(t) \\ x_2(t+1) &= 0.5x_2(t) + x_1(t) \\ x_3(t+1) &= 0.2x_3(t) + f(t) \end{aligned} \tag{11}$$

In matrix notations we have,

$$\begin{bmatrix} X_1(t+1) \\ X_2(t+1) \\ X_3(t+1) \end{bmatrix} = \begin{bmatrix} 0.5 & 0 & 0 \\ 1 & 0.5 & 0 \\ 0 & 0 & 0.2 \end{bmatrix} \begin{bmatrix} X_1(t) \\ X_2(t) \\ X_3(t) \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} f(t)$$

and,

$$y(t) = \begin{bmatrix} 1.667 & 1.25 & 0.533 \end{bmatrix} \begin{bmatrix} X_1(t) \\ X_2(t) \\ X_3(t) \end{bmatrix} + [1]f(t) \tag{12}$$

This finally gives us,

$$A = \begin{bmatrix} 0.5 & 0 & 0 \\ 1 & 0.5 & 0 \\ 0 & 0 & 0.2 \end{bmatrix} \ \mathbf{b} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \ \mathbf{c}^T = \begin{bmatrix} 1.667 & 1.25 & 0.533 \end{bmatrix} \ d = 1. \tag{13}$$

(Moon and Stirling, 1.4.31)

If $y[t]$ has two real sinusoids,

$$y[t] = A\cos(\omega_1 t + \theta_1) + B\cos(\omega_2 t + \theta_2),$$

and the frequencies are know, determine a means of computing the amplitudes and phases from measurements at time instants $t_1,\ t_2, ...,\ t_N$.

Solution:

$$\begin{aligned}
y[t] &= A\cos(\omega_1 t + \theta_1) + B\cos(\omega_2 t + \theta_2) \\
y[t] &= A\left[\cos\omega_1 t.\cos\theta_1 - \sin\omega_1 t.\sin\theta_1\right] + B\left[\cos\omega_2 t.\cos\theta_2 - \cos\omega_2 t.\sin\theta_2\right] \\
&= A\cos\theta_1(\cos\omega_1 t) + A\sin\theta_1(\sin\omega_1 t) + B\cos\theta_2(\cos\omega_2 t) + B\sin\theta_2(\sin\omega_2 t). \qquad (14)
\end{aligned}$$

Now, for simplicity we consider the following variables,

$$\begin{aligned}
x &= A\cos\theta_1 \\
y &= -A\sin\theta_1 \\
z &= B\cos\theta_2 \\
w &= -B\sin\theta_2 \qquad (15)
\end{aligned}$$

This gives us,

$$y[t] = x(\cos\omega_1 t) + y(\sin\omega_1 t) + z(\cos\omega_2 t) + w(\sin\omega_2 t) \qquad (16)$$

The above equation for $t = t_1, t_2, \cdots, t_N$ can be written in matrix form as

$$\underbrace{\begin{bmatrix} \cos\omega_1 t_1 & \sin\omega_1 t_1 & \cos\omega_2 t_2 & \sin\omega_2 t_2 \\ \cos\omega_1 t_2 & \sin\omega_1 t_2 & \cos\omega_2 t_2 & \sin\omega_2 t_2 \\ \vdots & \vdots & \vdots & \vdots \\ \cos\omega_1 t_N & \sin\omega_1 t_N & \cos\omega_2 t_N & \sin\omega_2 t_N \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}}_{\underline{x}} = \underbrace{\begin{bmatrix} y[t_1] \\ y[t_2] \\ \vdots \\ y[t_N] \end{bmatrix}}_{\underline{y}} \qquad (17)$$

$$\mathbf{A}\underline{x} = \underline{y}. \qquad (18)$$

The above set of linear equations can be solved by inverting the matrix $\mathbf{A}$. If $N > 4$, and the measurements are noiseless, we can choose any four time instances such that the matrix $\mathbf{A}$ is invertible and $\underline{x}$ can be solved as $\underline{x} = \mathbf{A}^{-1}\underline{y}$. If $N > 4$ and the measurements contain noise, we can use pseudo inverse of $\mathbf{A}$ to obtain a MMSE estimate of $\underline{x}$ given by $\underline{x} = \left(\mathbf{A}\mathbf{A}^T\right)^{-1}\mathbf{A}^T\underline{y}$.

To get A, B, θ_1 and θ_2 we just substitute the following,

$$\begin{aligned}
A &= \sqrt{x^2 + y^2} \\
B &= \sqrt{z^2 + w^2} \\
\theta_1 &= \tan^{-1}\left(\frac{-y}{x}\right) + \tan^{-1}\left(\frac{-w}{z}\right) \qquad (19)
\end{aligned}$$

Problem 2

Vectors belonging to $\mathbb{R}^2$ are jointly distributed uniformly on a rhombus whose vertices are $(\pm A,\, 0)$ and $(0,\, \pm A)$. Obtain the marginal densities. Examine if the random variables are (a) statistically independent (b) correlated?

Solution:

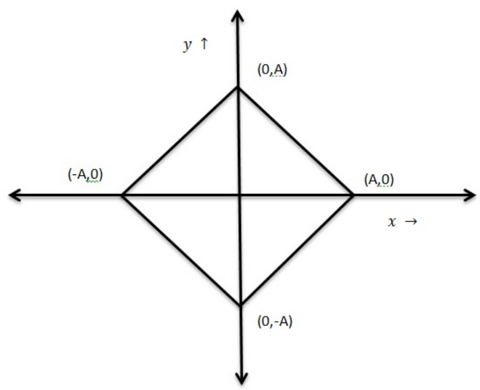


Figure 4

The joint density of the uniform random variable mentioned here is

$$\begin{aligned} f_{XY}(x,y) &= \frac{1}{\text{area of rhombus}} \\ &= \frac{1}{2A^2}, \end{aligned} \tag{20}$$

The marginal density along a particular axis is obtained by integrating the joint density with respect to the other axis. Calculating the marginals:

$$f_X(x) = \begin{cases} \displaystyle\int_{-(x+A)}^{x+A} \frac{1}{2A^2} dy, & -A \le x \le 0 \\ \displaystyle\int_{-(A-x)}^{A-x} \frac{1}{2A^2} dy, & 0 < x \le A \end{cases}$$

Solving the integral, we get

$$f_X(x) = \begin{cases} \frac{x+A}{A^2} & \text{for } x < 0. \\ \frac{A-X}{A^2} & \text{for } x \ge 0. \\ 0 & \text{otherwise.} \end{cases} \tag{21}$$

Similarly,

$$f_Y(y) = \begin{cases} \int_{-(y+A)}^{y+A} \frac{1}{2A^2} dx & -A \le y \le 0 \\ \int_{-(A-y)}^{A-y} \frac{1}{2A^2} dx & 0 < y \le A \end{cases}$$

which gives us,

$$f_Y(y) = \begin{cases} \frac{y+A}{A^2} & \text{for } y < 0. \\ \frac{A-y}{A^2} & \text{for } y \ge 0. \end{cases} \tag{22}$$

It is quite clear that $f_{X,Y}(x,y) \neq f_X(x) f_Y(y)$. Therefore, the random variables are not independent.

To determine correlation we calculate the covariance,

$$\mathrm{Cov}(X,Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y] \tag{23}$$

Calculating the expectations,

$$\begin{aligned}
\mathbb{E}[XY] &= \int\limits_{-A}^{0} \underbrace{\left(\int\limits_{-(A+x)}^{(A+x)} \frac{x}{2A^2} dx \right)}_{\text{odd function in } x} ydy + \int\limits_{0}^{A} \underbrace{\left(\int\limits_{-(A+x)}^{(A+x)} \frac{x}{2A^2} dx \right)}_{\text{odd function in } x} ydy \\
&= \int\limits_{-A}^{0} 0 \times ydy + \int\limits_{0}^{A} 0 \times ydy && (24) \\
\mathbb{E}[XY] &= 0 && (25)
\end{aligned}$$

$$\begin{aligned}
\mathbb{E}[X] &= \frac{1}{A^2}\int_{-A}^{0} x(A+x)dx + \frac{1}{A^2}\int_{0}^{A} x(A-x)dx \\
&= \frac{1}{A}\int_{-A}^{A} xdx + \frac{1}{A^2}\int_{-A}^{0} x^2 dx - \frac{1}{A^2}\int_{0}^{A} x^2 dx \\
&= 0 + \frac{1}{A^2}\frac{A^3}{3} - \frac{1}{A^2}\frac{A^3}{3} \\
&= 0.
\end{aligned}$$

Thus,

$$\mathrm{Cov}(X,Y) = 0,$$

which makes the random variables uncorrelated.

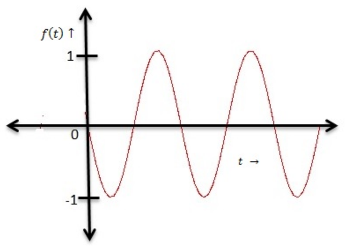
Problem 3

Consider a random process $Y(t) = Asin(\omega t)$ where A is a random variable uniformly distributed between $[-1, 1]$. Sketch the sample functions and obtain the probability distribution and cumulative distribution functions for the time instants $t = 0, \frac{\pi}{4\omega} \frac{\pi}{2\omega}$.

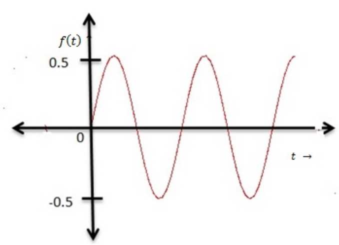
Solution:

$$y(t) = Asin\omega t.$$

A is a random variable distributed uniformly between [-1,1]. We have, for different values of A within the interval the following plots,



(a) $A = -1$



(b) $A = 0.5$

Figure 5

Calculating the probability distribution and cumulative distribution functions:

- $t = 0$.

  We get,

$$y(t) = 0.$$

  Thus, $y = 0$ always for whatever the values of A.

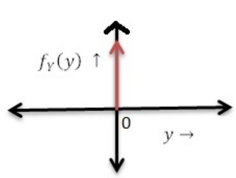


Figure 6: pdf of y

The cumulative distribution

$$\int_{-\infty}^{y} \delta(y)dy \quad = \quad F(Y \leq y) = \begin{cases} 0 & \text{for } y \leq 0^- \\ 1 & \text{for } y \geq 0^+ \end{cases}$$

Thus, this is a constant function for $t \geq 0$.

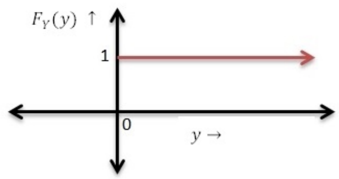


Figure 7: cdf of y

- $t = \frac{\pi}{4\omega}$
  Clearly,

$$y(t) = \frac{A}{\sqrt{2}}$$

Thus, pdf of y varies between the interval $\left[-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right]$ with a constant amplitude of $\frac{1}{\sqrt{2}}$.

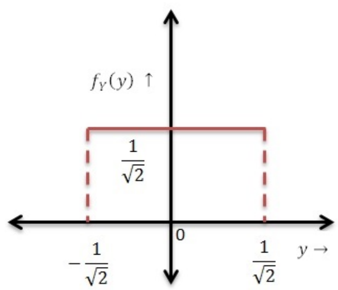


Figure 8: pdf of y

The cumulative distribution

$$\begin{aligned} F(Y \leq y) &= \int_{-\infty}^{y} f(y)dy \\ &= \int_{-\frac{1}{\sqrt{2}}}^{y} \frac{1}{\sqrt{2}}dy \\ &= \frac{y + \frac{1}{\sqrt{2}}}{\sqrt{2}}. \end{aligned}$$

for $y \in \left[-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right]$.

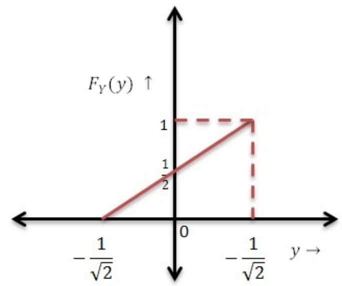


Figure 9: cdf of y

- $t = \frac{\pi}{2\omega}$

$$y(t) = A.$$

Thus, probability density function of y varies in [-1,1] with a constant magnitude of $\frac{1}{2}$.

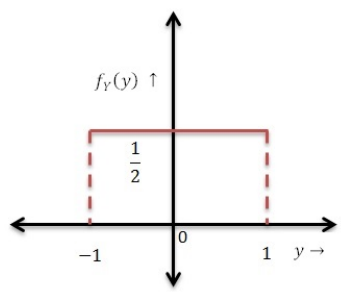


Figure 10: pdf of y

The cumulative distribution

$$\begin{aligned} F(Y \leq y) &= \int_{-\infty}^{y} f(y)dy \\ &= \int_{-1}^{y} \frac{1}{2} dy \\ &= \frac{y+1}{2}. \end{aligned}$$

for $y \in [-1, 1]$.

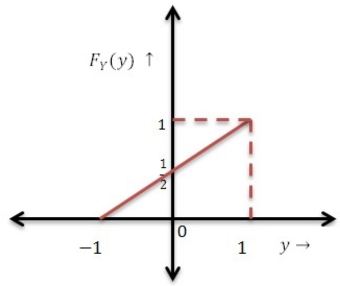


Figure 11: cdf of y

Problem 4

Sketch the regions in $\mathbb{R}^2$ for all vectors whose $\mathbb{L}3$ and $\mathbb{L}4$ norms are less than or equal to unity.

Solution:

$\mathbb{L}3$ norm $\leq 1 \implies \left(|x|^3 + |y|^3\right)^{\frac{1}{3}} \leq 1 \implies |x|^3 + |y|^3 \leq 1.$

$\mathbb{L}4$ norm $\leq 1 \implies \left(|x|^4 + |y|^4\right)^{\frac{1}{4}} \leq 1 \implies |x|^4 + |y|^4 \leq 1.$

The boundaries of the two regions given by $|x|^3 + |y|^3 = 1$ and $|x|^4 + |y|^4 = 1$ are plotted in the following figure.

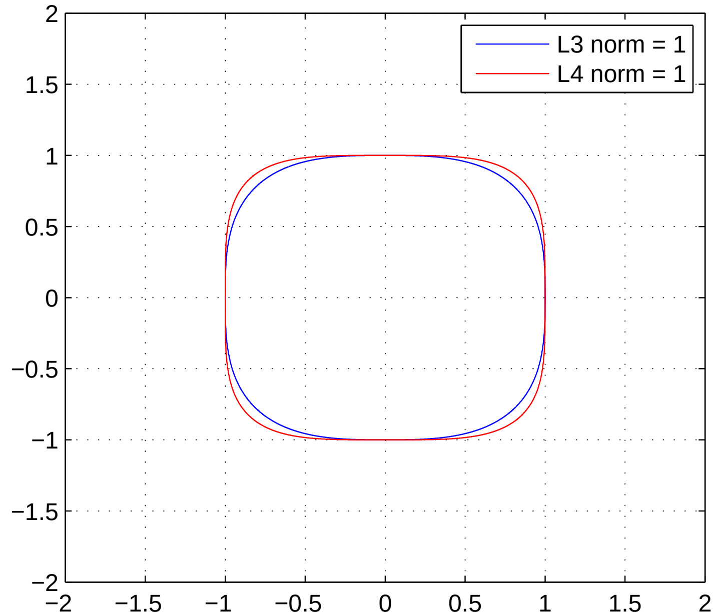


Figure 12

Problem 5 (Moon and Stirling, 2.2.28)

Let S be a finite dimensional vector space with $\dim(S) = m$. Show that every set containing $m+1$ points is linearly dependent.

Solution:

Since $\dim(S) = m$, we can find a Hamel basis B with m vectors.

Let us assume that there exists a set V of $m+1$ vectors that are linearly independent. We have two cases: 1) $\mathrm{Span}\,(V) = S$: V is a Hamel basis for S 2) $\mathrm{Span}\,(V) \neq S$: In this case we can find a Hamel basis W such that $V \subset W$.

Therefore, we can find a Hamel basis W such that $V \subseteq W$. From the Theorem proved in the class, the cardinalities of any two Hamel basis for a vector space must be the same:

$$\text{i.e., } |B| = |W|$$

But $|B| = m$ and $|W| \geq |V| = m+1$, a contradiction.

Therefore any set of $m+1$ vectors must be linearly dependent.

Alternate proof:

Let us assume for the finite dimensional vector space S the set A=$\{a_1, a_2, ..., a_m\}$ be the basis. Also consider the set $B = \{b_1, b_2, ..., b_{m+1}\}$ taken from the same space. It should be noted that none of the vectors from B are lying in A.

Let us assume at first that the set B is a linearly independent set. This means

$$\sum_{i=1}^{m+1} k_i b_i = 0$$

and the only solution is $k_i = 0$ for $1 \leq i \leq m+1$. Since the set A is a basis we will also have,

$$b_k = \sum_{i=1}^{m} \alpha_{1,i} a_i$$

for $1 \leq k \leq m+1$. Assume $\alpha_{1,1} \neq 0$, then,

$$a_1 = \frac{b_1}{\alpha_{1,1}} - \sum_{i=2}^{m} \frac{\alpha_{1,i}}{\alpha_{1,1}} a_i$$

also for any $c \in S$,

$$c = \sum_{i=1}^{m} l_i a_i = \sum_{i=2}^{m} l_i a_i + l_1 a_1$$

Substituting $a_1$we get,

$$\begin{aligned} c &= \sum_{i=2}^{m} l_i a_i + l_1 \left(\frac{b_1}{\alpha_{1,1}} - \sum_{i=2}^{m} \frac{\alpha_{1,i}}{\alpha_{1,1}} a_i \right) \\ &= l_1 \frac{b_1}{\alpha_{1,1}} - \sum_{i=2}^{m} \left(l_i - \frac{l_1 \alpha_{1,i}}{\alpha_{11}} \right) a_i. \end{aligned}$$

Thus, any c can be represented in terms of a_i for $2 \leq i \leq m$ and b_1 This shows that the set $\{b_1, a_2, a_3, ..., a_m\}$ spans the whole space. Similarly, we can replace a_k with b_k for $2 \leq k \leq m$ which will eventually make the set $\{b_1, b_2, ..., b_m\}$ as a Hamel basis for S i.e., b_{m+1} can be generated from the set $\{b_1, b_2, ..., b_m\}$. Hence for a m-dimensional space every set containing $m+1$ vectors is linearly dependent.

(Moon and Stirling, 2.2.33)

Show that in a normed linear space,

$$| \; ||x|| - ||y|| \; | \leq ||x - y||.$$

Solution:

Let S be a normed vector space and also $x, y \in S$. Since be have assumed S to be a vector space, we will also have the vector $(x - y) \in S$. Now for any two vector $p, q \in S$ the following identity is known to be true.

$$||p + q|| \leq ||p|| + ||q|| \tag{26}$$

if we set, $p = y$ and $q = x - y$, then

$$\begin{aligned} ||y + x - y|| &\leq ||y|| + ||x - y|| \\ \text{or, } ||x|| - ||y|| &\leq ||x - y||. \end{aligned} \tag{27}$$

similarly if we set $p = x$ and $q = y - x$, then

$$\begin{aligned} ||x + y - x|| &\leq ||x|| + ||y - x|| \\ \text{or, } ||y|| - ||x|| &\leq ||x - y|| \end{aligned} \tag{28}$$

Since $||y - x|| = || - (x - y)|| = ||x - y||$. Thus from the last two equations,

$$| \; ||x|| - ||y|| \; | \leq ||x - y||. \tag{29}$$

(Moon and Stirling 2.2.32)

Show that the set 1, t, ... , t^m is a linearly independent set.

Solution:

Consider the set of all polynomials of degree m or less. Let us assume that the set $\{1, t, \ ... \ ,t^m\}$ is linearly dependent. According to our assumption we get,

$$\alpha_1 + \alpha_2 t + ... + \alpha_{m+1} t^m = 0,$$

where, at least one of α_i for $1 \leq i \leq m+1$ is non zero and $a_{m+1} \neq 0$. The above equation is true for any value of t. Hence, the above equation has infinite solutions. But according to the fundamental theorem of algebra, the above equation can have exactly m roots which leads to a contradiction. Hence the set $\{1, \text{t}, \ ... \ ,t^m\}$ is linearly independent.

Alternate proof:

Let us try to identify $\alpha_1, \alpha_2, \cdots, \alpha_{m+1}$ such that the following equation is always true,

$$\alpha_1 + \alpha_2 t + ... + \alpha_{m+1} t^m = 0.$$

where, at least one of α_i for $1 \leq i \leq m+1$ is non zero and $a_{m+1} \neq 0$. Substituting $t = 1, b, b^2, b^3 \cdots b^m$ in the above equation, we get the following set of equations:

$$\underbrace{\begin{bmatrix} 1 & b & b^2 & \cdots & b^m \\ 1 & b^2 & b^4 & \cdots & b^{2m} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & b^m & b^{2m} & \cdots & b^{m^2} \end{bmatrix}}_{B} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_{m+1} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

The matrix B is a Vandermonde matrix which has full rank. Therefore, we can invert B to obtain α_is as

$$\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_{m+1} \end{bmatrix} = B^{-1} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Therefore, the only values of $\alpha_1, \alpha_2, \cdots, \alpha_{m+1}$ is all zeros. Hence the set $\{1, \text{t}, \ ... \ ,t^m\}$ is linearly independent.