Indian Institute of Science

E9-252: Mathematical Methods and Techniques in Signal Processing

Instructor: Shayan Srinivasa Garani

Mid Term Exam#2, Fall 2016

Name and SR.No:

Instructions:

- You are allowed only 4 pages of written notes and a calculator for this exam. No wireless allowed.
- The time duration is 3 hrs.
- There are four main questions. None of them have negative marking.
- Attempt all of them with careful reasoning and justification for partial credit.
- Do not panic, do not cheat.
- Good luck!

Question No.	Points scored
1	
2	
3	
4	
Total points	

SOLUTIONS:

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PROBLEM 1: This problem has 3 parts.
Let $\mathcal{V}_n$ be the spaces generated from the Haar scaling function $\phi(2^n t - k)$, where k is any integer.

(1) What are the dimensions of $\mathcal{V}_n$ and $\mathcal{W}_n$ over the interval $t \in [0, 1)$? (5 pts.)

(2) With the usual notations as followed in the class, we have $\mathcal{V}_n = \mathcal{V}_0 \bigoplus_{k=0}^{n-1} \mathcal{W}_k$. Obtain the dimensions in $\mathcal{V}_0$ and $\{\mathcal{W}_k\}_{k=0}^{n-1}$ and use this to evaluate your answer for the dimension of $\mathcal{V}_n$. How does it compare to your result in sub part 1 of Problem 1? (10 pts.)

(3) Expand the signal $s(t) = 1 - t^2$ over the interval $t \in [-1, 1]$ using the Haar wavelet. (10 pts.)

Solution

Part 1: The wavelets in V_n and W_n have support over non-overlapping intervals of length 2^{-n}. Therefore, the dimensions of V_n and W_n over the interval $t \in [0, 1)$ is 2^n.

$$\dim(V_n) = 2^n = \dim(W_n).$$

Part 2: $V_0, W_0, W_1, ...$ are all orthogonal spaces. Therefore,

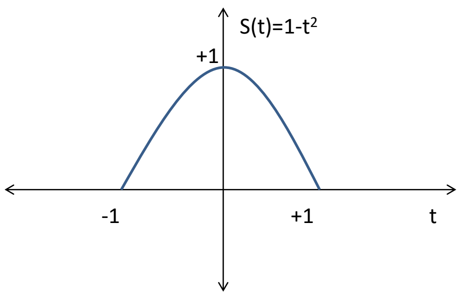
$$V_n = V_0 \bigoplus_{k=0}^{n-1} W_k$$

$$\implies \dim(V_n) = \dim(V_0) + \sum_{k=0}^{n-1} \dim(W_K)$$

$$= 1 + \sum_{k=0}^{n-1} 2^k = 1 + (2^n - 1) = 2^n.$$

The result agrees with part 1.

Part 3:



Since the support of $s(t)$ is $[-1, 1]$, the wavelet representation corresponding to W_n will have 2^{n+1}coefficients.

$$s(t) = a_{-1}^{(0)}\phi(t+1) + a_0^{(0)}\phi(t) + \sum_{n=0}^{\infty} \sum_{k=-2^n}^{2^n-1} b_k^n \psi(2^n t - k)$$

The coefficients $a_{-1}^{(0)}, a_0^{(0)}, \{b_k^{(n)}\}$ can be obtained by projecting $s(t)$ on to orthogonal basis.

$$\langle \phi(t+1), s(t) \rangle = a_{-1}^{(0)} \langle \phi(t+1), \phi(t+1) \rangle$$

$$\implies a_{-1}^{(0)} = \langle \phi(t+1), s(t) \rangle = \int_{-1}^{0} (1 - t^2) dt = \frac{2}{3} \implies a_{-1}^{(0)} = \frac{2}{3}.$$

Similarly,

$$a_0^{(0)} = \frac{2}{3}$$

$$\psi(2^n t-k)=\begin{cases}1, & \frac{k}{2^n}\le t\le \frac{k}{2^n}+\frac{1}{2^{n+1}}\\ -1, & \frac{k}{2^n}+\frac{1}{2^{n+1}}\le t\le \frac{k+1}{2^n}\\ 0, & \text{otherwise.}\end{cases}$$

$$\langle \psi(2^n t-k),\psi(2^n t-k)\rangle=\frac{1}{2^n}$$

$$\begin{aligned}\Longrightarrow b_k^n &= 2^n\left\langle s(t),\psi(2^n t-k)\right\rangle\\ &= 2^n\int_{\frac{k}{2^n}}^{\frac{k}{2^n}+\frac{1}{2^{n+1}}}(1-t^2)dt-\int_{\frac{k}{2^n}+\frac{1}{2^{n+1}}}^{\frac{k+1}{2^n}}(1-t^2)dt\\ &= 2^n[\frac{2k+1}{2^{n+1}}-\frac{1}{3}(\frac{2k+1}{2^{n+1}})^3-\frac{k}{2^n}+\frac{1}{3}\left(\frac{k}{2^n}\right)^3\\ &= (2k+1)2^{-2n-2}.\end{aligned}$$

$$\therefore s(t)=\frac{2}{3}\phi(t+1)+\frac{2}{3}\phi(t)+\sum_{n=0}^{\infty}\sum_{k=-2^n}^{2^n-1}(2k+1)2^{-2n-2}\psi(2^n t-k)$$

PROBLEM 2: The following points $(2,0)^T$, $(3,-1)^T$, $(2,-2)^T$ and $(1,-1)^T$ occur with probabilities 1/8, 1/8, 3/8 and 3/8 respectively.

(1) Obtain the KL representation of the points. (8 pts.)

(2) If you were to reduce the points to 1D, how would you optimally represent them? Sketch the new 1D points carefully. (6 pts.)

(3) What fraction of the energy is lost by doing a dimensionality reduction in the previous step? (5 pts.)

(4) Suppose these four points correspond to four different classes, sketch the linear decision boundaries to separate the original set of points in 2D as well as in 1D after dimensionality reduction. Write down the equations of the boundaries explicitly. Are the linear decision boundaries unique? (6 pts.)

Solution

Part 1: $\mu = \frac{1}{8}\begin{bmatrix} 2 \\ 0 \end{bmatrix} + \frac{1}{8}\begin{bmatrix} 3 \\ -1 \end{bmatrix} + \frac{3}{8}\begin{bmatrix} 2 \\ -2 \end{bmatrix} + \frac{3}{8}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} \frac{7}{4} \\ \frac{-5}{4} \end{bmatrix}$

$$
\begin{aligned}
C + \mu\mu^\top &= \begin{bmatrix} \frac{7}{2} & \frac{-9}{4} \\ \frac{-9}{4} & 2 \end{bmatrix} \\
\Longrightarrow C &= \begin{bmatrix} \frac{7}{2} & \frac{-9}{4} \\ \frac{-9}{4} & 2 \end{bmatrix} - \begin{bmatrix} \frac{7}{4} \\ \frac{-5}{4} \end{bmatrix} \begin{bmatrix} \frac{7}{4} & \frac{-5}{4} \end{bmatrix} \\
&= \frac{1}{16}\begin{bmatrix} 7 & -1 \\ -1 & 7 \end{bmatrix}.
\end{aligned}
$$

The eigen decomposition of the covariance matrix is:

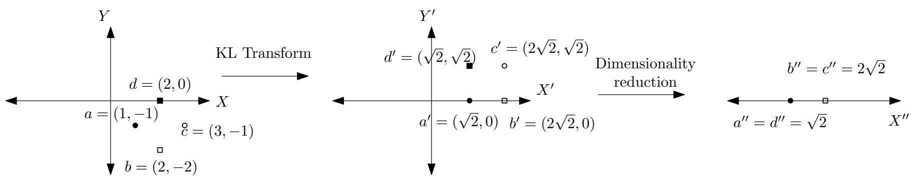
$$
C = \frac{1}{16}\begin{bmatrix} 7 & -1 \\ -1 & 7 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{3}{8} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}.
$$

The eigen values are $\lambda_1 = \frac{1}{2}, \lambda_2 = \frac{3}{8}$.

Therefore KL representation of the point $\begin{bmatrix} x \\ y \end{bmatrix}$ is $\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} x-y \\ x+y \end{bmatrix}$.

$\begin{bmatrix} 2 \\ 0 \end{bmatrix} \leftrightarrow \begin{bmatrix} \sqrt{2} \\ \sqrt{2} \end{bmatrix}; \begin{bmatrix} 3 \\ -1 \end{bmatrix} \leftrightarrow \begin{bmatrix} 2\sqrt{2} \\ \sqrt{2} \end{bmatrix}; \begin{bmatrix} 2 \\ -2 \end{bmatrix} \leftrightarrow \begin{bmatrix} 2\sqrt{2} \\ 0\sqrt{2} \end{bmatrix}; \begin{bmatrix} 1 \\ -1 \end{bmatrix} \leftrightarrow \begin{bmatrix} \sqrt{2} \\ 0 \end{bmatrix}$

Part 2: If the points are reduced to 1D, we take the first coordinate in the KL representation so that maximum energy is retained. The new points are
$a'' = \sqrt{2}$, $b'' = 2\sqrt{2}$, $c'' = 2\sqrt{2}$, $d'' = \sqrt{2}$.



Part 3: Fraction of energy lost with dimensionality reduction $= \frac{\lambda_2}{\lambda_1+\lambda_2} = \frac{\frac{3}{8}}{\frac{1}{2}+\frac{3}{8}} = \frac{3}{7}$.

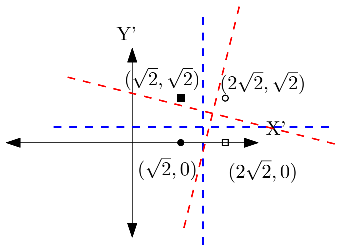
Part 4:

Since there is no noise, we can choose any decision boundaries that separate the four points. We choose following boundaries

$$x = 1.5; \quad y = -1.5.$$

The boundaries are not unique. A different choice of boundary is

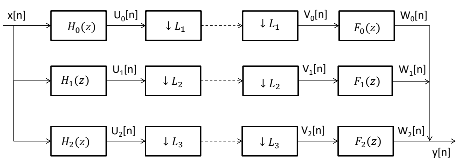
$$x = 1.25; \quad y = -1.25.$$



In the case of 1D, the four points cannot be classified without confusion because $a'' = d''$ and $b'' = c''$. The two points can be classified by the boundary $x = t$ for any $t \in (\sqrt{2}, 2\sqrt{2})$.

PROBLEM 3: Two students claimed to design a 3-channel perfect reconstruction filterbank. In one case, the student had a bank of analysis filters followed by down sampling rates of 3 in each branch. In another case, the down sampling rates were chosen to be 2, 3 and 6 in each of the three branches following the analysis filters. Justify if their claims are correct. Suppose a third student decided to go with a wavelet 3-channel filter bank using the Haar basis towards perfect reconstruction, what would you expect the down sampling rates to be at the analysis stage? Justify. (25 pts.)

Solution



If $f_3$is the sampling rate of $x[n]$, then the number of samples/sec at the output of analysis bank is,
$\frac{F_s}{L_1} + \frac{F_s}{L_2} + \frac{F_s}{L_3}$
There should be no sample rate loss to be able to reconstruct perfectly. This gives us a necessary condition on the decimation rates:

$$\frac{F_s}{L_1} + \frac{F_s}{L_2} + \frac{F_s}{L_3} \geq F_s$$
$$\Longrightarrow \frac{1}{L_1} + \frac{1}{L_2} + \frac{1}{L_3} \geq 1.$$

In both the cases $(L_{1,}L_2, L_3) = (3,3,3)$ and $(2,3,6)$, we have $\frac{1}{L_1} + \frac{1}{L_2} + \frac{1}{L_3} = 1$.

First student:

$(L_{1,}L_2, L_3) = (3,3,3)$ is already studied in the class and is known to be feasible for perfect reconstruction.

Second student:

$$U_i(z) = X(z) H_i(z), \quad i = 0, 1, 2.$$

$$V_0(z) = \frac{1}{2}\sum_{i=0}^{1} U_0\left(z\omega_2^i\right) = \frac{1}{2}\sum_{i=0}^{1} X\left(z\omega_2^i\right) H_0\left(z\omega_2^i\right), \quad \omega_2 = e^{j\pi}$$
$$V_1(z) = \frac{1}{3}\sum_{i=0}^{2} U_1\left(z\omega_3^i\right) = \frac{1}{3}\sum_{i=0}^{2} X\left(z\omega_3^i\right) H_1\left(z\omega_3^i\right), \quad \omega_3 = e^{j\frac{2\pi}{3}}$$
$$V_2(z) = \frac{1}{6}\sum_{i=0}^{5} U_2\left(z\omega_6^i\right) = \frac{1}{6}\sum_{i=0}^{5} X\left(z\omega_6^i\right) H_2\left(z\omega_6^i\right), \quad \omega_6 = e^{j\frac{2\pi}{6}}.$$

$$W_i(z) = V_i(z) F_i(z), \quad i = 0, 1, 2.$$

$$\begin{aligned} Y(z) &= \sum_{i=0}^{2} W_i(z). \\ &= \frac{1}{2} F_0(z) \sum_{i=0}^{1} X\left(z\omega_2^i\right) H_0\left(z\omega_2^i\right) \\ &+ \frac{1}{3} F_1(z) \sum_{i=0}^{2} X\left(z\omega_3^i\right) H_1\left(z\omega_2^i\right) \\ &+ \frac{1}{6} F_2(z) \sum_{i=0}^{3} X\left(z\omega_6^i\right) H_2\left(z\omega_2^i\right). \end{aligned}$$

We have $\omega_3 = \omega_6^2$ and $\omega_2 = \omega_6^3$. We can write $Y(z)$ using $X(z)$ and alias components $\left\{X\left(z\omega_6^k\right)\right\}_{k=1\cdots 5}$ as

$$Y(z) = \sum_{i=0}^{5} Y_i(z) X\left(z\omega_6^i\right),$$

where

$$\begin{bmatrix} Y_0(z) \\ Y_1(z) \\ Y_2(z) \\ Y_3(z) \\ Y_4(z) \\ Y_5(z) \end{bmatrix} = \begin{bmatrix} \frac{1}{2}H_0(z) & \frac{1}{3}H_1(z) & \frac{1}{6}H_2(z) \\ 0 & 0 & \frac{1}{6}H_2(z\omega_6) \\ 0 & \frac{1}{3}H_1\left(z\omega_6^2\right) & \frac{1}{6}H_2\left(z\omega_6^2\right) \\ \frac{1}{2}H_0\left(z\omega_6^3\right) & 0 & \frac{1}{6}H_2\left(z\omega_6^3\right) \\ 0 & \frac{1}{3}H_1\left(z\omega_6^4\right) & \frac{1}{6}H_2\left(z\omega_6^4\right) \\ 0 & 0 & \frac{1}{6}H_2\left(z\omega_6^5\right) \end{bmatrix} \begin{bmatrix} F_0(z) \\ F_1(z) \\ F_2(z) \end{bmatrix} \tag{1}$$

For perfect reconstruction, we require $Y_i(z) = 0, \quad i = 1,2,3,4,5$ and $Y_0(z) = 1$.

$$Y_1(z) = 0 \implies F_2(z) \times \frac{1}{6}H_2(z\omega_6) = 0 \implies F_2(z) = 0.$$

$$Y_2(z) = 0 \implies \frac{1}{3}F_1(z)H_1\left(z\omega_6^2\right) + \frac{1}{6}F_2(z)H_2\left(z\omega_6^2\right) = 0 \implies F_1(z) = 0$$

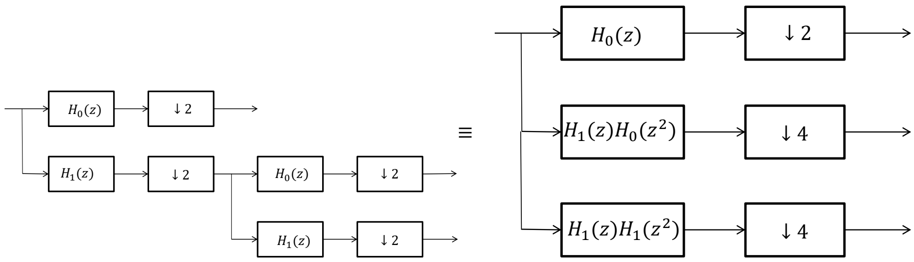
$$Y_3(z) = 0 \implies \frac{1}{2}F_0(z)H_0\left(z\omega_6^3\right) + \frac{1}{6}F_3(z)H_2\left(z\omega_6^3\right) = 0 \implies F_0(z) = 0.$$

However, this results in $Y_0(z) = 1$. Therefore, the set of equations in (1) are inconsistent.

Therefore, decimation rates of $(3,3,3)$ is feasible while $(2,3,6)$ is not feasible for perfect reconstruction.

Third student:

Wavelet 3-channel analysis bank is



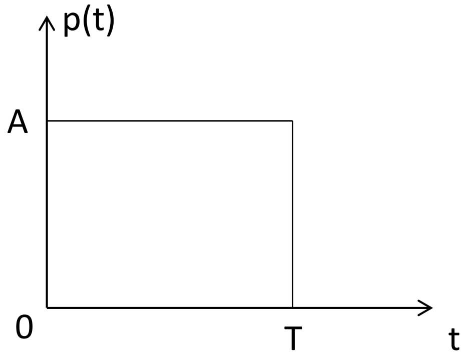
$H_0(z) = \frac{1-z^{-1}}{\sqrt{2}}$

$H_1(z) = \frac{1+z}{\sqrt{2}}$

The down sampling rates are 2,4,4.

PROBLEM 4: Consider a rectangular pulse $p(t)$ of amplitude A starting at the origin having a duration of T seconds. Treating time and frequency as random variables, compute the mean in time μ_t and in frequency μ_ω using a measure of the induced norm of the signal as a distribution. Justify if $\mu_t \leq \mu_\omega$ is true or false. (25 pts.)

Solution



Given

$$p(t) = \begin{cases} A, & 0 \leq t \leq T \\ 0, & \text{otherwise.} \end{cases}$$

Fourier transform of a rectangular pulse is a sinc pulse.

$$r(t) = \begin{cases} 1, & -\frac{1}{2} \leq t \leq \frac{1}{2} \\ 0, & \text{otherwise} \end{cases} \implies p(t) = Ar\left(\frac{t}{T} - \frac{1}{2}\right).$$

$$\begin{aligned} \mathcal{F}[r(t)] &= \text{sinc}(\omega) \\ \implies \mathcal{F}\left[r\left(t - \frac{1}{2}\right)\right] &= e^{j\frac{\omega}{2}} \text{sinc}(\omega) \\ \implies \mathcal{F}\left[Ar\left(\frac{t}{T} - \frac{1}{2}\right)\right] &= Ae^{j\frac{\omega T}{2}} \text{sinc}(\omega T) \end{aligned}$$

Therefore, Fourier transform of $p(t)$ is

$$P(\omega) = Ae^{-j\omega\frac{T}{2}} \text{sinc}(\omega T)$$

p.d.f in time domain is obtained by normalizing the time-domain signal $p(t)$ as given by

$$f_t(t) = \frac{1}{AT} p(t) = \begin{cases} \frac{1}{T}, & 0 \leq t \leq T, \\ 0, & \text{otherwise.} \end{cases}$$

$$\implies \mu_t = \int_0^T \frac{1}{T} t dt = \frac{T}{2}.$$

Similarly, p.d.f distribution in frequency domain is

$$f_\omega(\omega) = k \mid A\text{sinc}(\omega T) \mid = \begin{cases} k\left|A\frac{\sin(\omega T)}{\omega T}\right|, & \omega \neq 0, \\ k\left|A\right|, & \omega = 0 \end{cases}$$

for some constant k such that $\int_{-\infty}^{\infty} f_\omega(\omega)\, d\omega = 0$.

Since $f_\omega(-\omega) = f_\omega(\omega)$, $f_\omega(\omega)\,\omega$ is an odd function. Therefore the mean of frequency is,

$$\mu_\omega = \int_{-\infty}^{\infty} f_\omega(\omega)\,\omega d\omega = 0.$$

Therefore $\mu_t > \mu_\omega$.