INDIAN INSTITUTE OF SCIENCE
E9-252: MATHEMATICAL METHODS AND TECHNIQUES IN SIGNAL PROCESSING
HOME WORK #3 - SOLUTIONS, FALL 2015

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Problem 1. If $\mathcal{V}$ and $\mathcal{W}$ are finite dimensional orthogonal subspaces of an inner product space $\mathcal{H}$, prove that $\dim(\mathcal{V} \oplus \mathcal{W}) = \dim(\mathcal{V}) + \dim(\mathcal{W})$.

Solution. Let $\dim(\mathcal{V}) = d_v$ and $\dim(\mathcal{W}) = d_w$. There exists a set of d_v vectors $\left\{\underline{v}_1, \underline{v}_2, \cdots, \underline{v}_{d_v}\right\}$ that form an orthogonal basis for $\mathcal{V}$. Similarly, we can find a set of d_w vectors $\left\{\underline{w}_1, \underline{w}_2, \cdots, \underline{w}_{d_v}\right\}$ that form an orthogonal basis for $\mathcal{W}$. Since the $\mathcal{V}$ and $\mathcal{W}$ are orthogonal subspaces, the orthogonal basis vectors to $\left\{\underline{v}_1, \underline{v}_2, \cdots, \underline{v}_{d_v}, \underline{w}_1, \underline{w}_2, \cdots, \underline{w}_{d_v}\right\}$ are all orthogonal to each other. Therefore, they all are linearly independent.

Any vector in $\mathcal{V} \oplus \mathcal{W}$ can be represented as a linear combination of two vectors $\underline{v}$ and $\underline{w}$ where $\underline{v} \in \mathcal{V}$ and $\underline{w} \in \mathcal{V}$. Since any vector in each of the subspaces can be represented as a linear combination of orthogonal basis vectors, any vector in $\mathcal{V} \oplus \mathcal{W}$ can be represented as a linear combination of vectors in $\left\{\underline{v}_1, \underline{v}_2, \cdots, \underline{v}_{d_v}, \underline{w}_1, \underline{w}_2, \cdots, \underline{w}_{d_v}\right\}$. Therefore, the set of vectors $\left\{\underline{v}_1, \underline{v}_2, \cdots, \underline{v}_{d_v}, \underline{w}_1, \underline{w}_2, \cdots, \underline{w}_{d_v}\right\}$ form an orthogonal basis for $\mathcal{V} \oplus \mathcal{W}$.

Therefore,

$$\dim(\mathcal{V} \oplus \mathcal{W}) = d_v + d_w = \dim(\mathcal{V}) + \dim(\mathcal{W}).$$

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Problem 2. Obtain the Haar wavelet decomposition of the signal $f(t)$. Indicate the signal dimension at each subspace carefully. Devise a generic algorithm for doing Haar decomposition using a computer program.

$$f(t) = \begin{cases} 2 & -2 \leq t < -1 \\ -4 & -1 \leq t < -0.5 \\ -2 & -0.5 \leq t < 0 \\ 2 & 0 \leq t < 0.25 \\ 1 & 0.25 \leq t \leq 2 \end{cases}$$

Solution. The Haar wavelet decomposition of the signal is $\underline{a}^{(0)} = (2, -3, 1.25, 1)$, $\underline{b}_0 = (0, -1, 0.25, 0)$, $\underline{b}_1 = (0, 0, 0, 0, 0.5, 0, 0, 0)$.

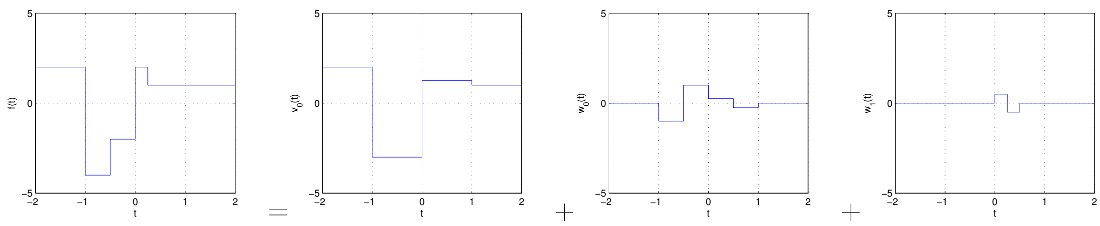
Dimension of the signal in $\mathcal{V}_0$ is 4, in $\mathcal{V}_1$ is 8 and in $\mathcal{V}_2$ is 16. The coefficients of Haar wavelet and scaling function in different subspaces are given below.

$a_k^{(j)}$	$k=-8$	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7
$j=2$	2	2	2	2	-4	-4	-2	-2	2	1	1	1	1	1	1	1
$j=1$					2	-4	-2	-2	1.5	1	1	1				
$j=0$							2	-3	1.25	1						

$b_k^{(j)}$	$k=-4$	-3	-2	-1	0	1	2	3
$j=1$	0	0	0	0	0.5	0	0	0
$j=0$			0	-1	0.25	0		

TABLE 1. Coefficients of Haar wavelets and scaling function in different subspaces.

Figure 1 shows the decomposition of $f(t)$ as $v_0(t) + w_0(t) + w_1(t)$.



(A) $f(t)$ (B) $v_0(t)$ (C) $w_0(t)$ (D) $w_1(t)$

FIGURE 1. Haar decomposition of $f(t)$ as $v_0(t) + w_0(t) + w_1(t)$.

MATLAB code for wavelet decomposition is given below. The code uses the following relations that were derived in the class:

$$a_k^{(j-1)} = \frac{a_{2k}^{(j)} + a_{2k+1}^{(j)}}{2}$$

$$b_k^{(j-1)} = \frac{a_{2k}^{(j)} - a_{2k+1}^{(j)}}{2}$$

```
%% Signal Representation and Properties %%
intervals = [   -2,    -1,  -0.5,    0,  0.25,   2];
vals      = [    2,    -4,    -2,    2,         1];

num_intervals        = length(intervals) - 1;
resolution           = 0.25;
duration             = intervals(end) - intervals(1);
num_decompositions   = -log2(resolution);
signal_dim           = duration/resolution;

space_resolution = (0.5).^(0:num_decompositions);
a = zeros(num_decompositions + 1, signal_dim);
b = zeros(num_decompositions, signal_dim);
```

```matlab
%% Signal representation using Haar scaling function %%
j = 1;
for i=1:(signal_dim)
    end_time = intervals(1) + resolution*i;
    if end_time <= intervals(j+1)
        a(1, i) = vals(j);
    else
        j = j + 1;
        a(1, i) = vals(j);
    end
end

a(1,end) = vals(end);

%% Haar Wavelet Decomposition %%
for i=1:(num_decompositions)
    a(i+1,1:(signal_dim/2)) = (a(i,1:2:signal_dim) + a(i, 2:2:signal_dim))/2;
    b(i,1:(signal_dim/2))   = (a(i,1:2:signal_dim) - a(i, 2:2:signal_dim))/2;
end


%% Plotting the projections %%
space_resolution = resolution;
space_dim        = signal_dim;
for i=1:(num_decompositions+1)
    X = zeros(1,space_dim*2);
    Y = zeros(1,space_dim*2);
    X(1:2:end) = intervals(1) + (0:(space_dim-1))*space_resolution;
    X(2:2:end) = intervals(1) + (1:space_dim)*space_resolution;
    Y(1:2:end) = a(i, 1:space_dim);
    Y(2:2:end) = a(i, 1:space_dim);
    plot(X,Y); grid on; ylim([-5,5]); figure;
    space_resolution = 2*space_resolution;
    space_dim = space_dim/2;
end


%% Plotting the wavelet decompositions %%
space_resolution = resolution;
space_dim        = signal_dim/2;
for i=1:(num_decompositions)
    X = zeros(1,space_dim*4);
    Y = zeros(1,space_dim*4);
    X(1:2:end) = intervals(1) + (0:(2*space_dim-1))*space_resolution;
    X(2:2:end) = intervals(1) + (1:(2*space_dim))*space_resolution;
    Y(1:4:end) = b(i, 1:space_dim);
    Y(2:4:end) = b(i, 1:space_dim);
    Y(3:4:end) = -b(i, 1:space_dim);
    Y(4:4:end) = -b(i, 1:space_dim);
    plot(X,Y); grid on; ylim([-5,5]); figure;
    space_resolution = 2*space_resolution;
    space_dim = space_dim/2;
end
```

Problem 3. Prove the following properties for Haar wavelets:

- Parseval's equality i.e., energy conservation relation.
- Orthogonality across scales and time translates.

Solution. Orthogonality across scales and time translates

We have the Haar wavelet

$$\psi(t) = \begin{cases} 1, & 0 \le t < \frac{1}{2} \\ -1, & \frac{1}{2} \le t < 1 \\ 0 & \text{otherwise.} \end{cases}$$

The scaled and shifted versions are

$$\psi\left(2^j t - k\right) = \begin{cases} 1, & k2^{-j} \le t < \left(k + \frac{1}{2}\right) 2^{-j} \\ -1, & \left(k + \frac{1}{2}\right) 2^{-j} \le t < (k+1)\, 2^{-j} \\ 0 & \text{otherwise.} \end{cases}$$

Shift orthogonality: For $k \neq l$, $\left[k2^{-j}, (k+1)\, 2^{-j}\right)$ and $\left[l2^{-j}, (l+1)\, 2^{-j}\right)$ are non-overlapping regions. Therefore,

$$\left\langle \psi\left(2^j t - k\right), \psi\left(2^j t - l\right) \right\rangle = 0. \quad k \neq l$$

$$\left\langle \psi\left(2^j t - k\right), \psi\left(2^j t - k\right) \right\rangle = \int_{k2^{-j}}^{(k+1)2^{-j}} 1 dt = 2^{-j}.$$

Orthogonality across scales: Without loss of generality let, $p > q$ be two different scales. Therefore, $\psi\left(2^p t - k\right)$ has a smaller support than $\psi\left(2^q t - l\right)$ i.e., $2^{-p} < 2^{-q}$. Notice

1) $\psi\left(2^q t - l\right)$ is constant in each of the intervals $\left(m2^{-(q+1)}, (m+1)\, 2^{-(q+1)}\right), m \in \mathbb{Z}$.

2) For $p > q$, any interval $\left(n2^{-p}, (n+1)\, 2^{-p}\right), n \in \mathbb{Z}$ is a proper subset of the interval $\left(m2^{-(q+1)}, (m+1)\, 2^{-(q+1)}\right)$ where $m = \left\lfloor n2^{q+1-p} \right\rfloor$.

Therefore,

$$\int_{k2^{-p}}^{\left(k+\frac{1}{2}\right)2^{-p}} \psi\left(2^p t - k\right) \psi\left(2^q t - l\right) dt = - \int_{\left(k+\frac{1}{2}\right)2^{-p}}^{(k+1)2^{-p}} \psi\left(2^p t - k\right) \psi\left(2^q t - l\right) dt$$

$$\begin{aligned} \Longrightarrow \left\langle \psi\left(2^p t - k\right), \psi\left(2^q t - l\right) \right\rangle &= \int_{k2^{-p}}^{\left(k+\frac{1}{2}\right)2^{-p}} \psi\left(2^q t - l\right) dt - \int_{\left(k+\frac{1}{2}\right)2^{-p}}^{(k+1)2^{-p}} \psi\left(2^q t - l\right) dt \\ &= 0 \end{aligned}$$

Therefore, the scales and time translates of Haar wavelets $\left\{\psi\left(2^j - k\right), k \in \mathbb{Z}, j = 0, 1, \cdots\right\}$ are all orthogonal to each other.

Parseval's equality

Let $f(t) \in L^1(\mathbb{R})$. From the class notes, the wavelet decomposition of the function is given by

$$f(t) = v_0(t) + \sum_{j=0}^{\infty} w_j(t), \tag{1}$$

where

$$\begin{aligned} v_0(t) &= \sum_{k=-\infty}^{\infty} a_k^{(0)} \phi(t-k) \in \mathcal{V}_0 \\ v_j(t) &= \sum_{k=-\infty}^{\infty} b_k^{(j)} 2^{j/2} \psi\left(2^j t - k\right) \in \mathcal{W}_j \end{aligned}$$

and the coefficients $a_k^{(0)}$ and $b_k^{(j)}$ are obtained by projecting $f(t)$ onto the orthonormal basis of $\mathcal{V}_0, \mathcal{W}_0, \mathcal{W}_1, \mathcal{W}_2 \cdots$, i.e.,

$$\begin{aligned} a_k^{(0)} &= \langle f(t), \phi(t-k) \rangle, \\ b_k^{(k)} &= \left\langle f(t), 2^{j/2} \psi\left(2^j t - k\right) \right\rangle. \end{aligned}$$

Our aim is to prove that the energy in the signal is equal to the sum of square of the wavelet coefficients $\left\{a_k^{(0)}, b_k^{(k)}, k \in \mathbb{Z},\ j = 0, 1, \cdots\right\}$ i.e.,

$$\int_{-\infty}^{\infty} |f(t)|^2\, dt = \sum_{k=-\infty}^{\infty} \left|a_k^{(0)}\right|^2 + \sum_{j=0}^{\infty} \sum_{k=-\infty}^{\infty} \left|b_k^{(j)}\right|^2$$

The subspaces $\mathcal{V}_0, \mathcal{W}_0, \mathcal{W}_1, \cdots$ are all orthogonal to each other. Therefore,

$$\begin{aligned} \langle v_0(t), w_j(t)\rangle &= 0, \quad j = 0, 1, 2 \cdots \\ \langle w_j(t), w_k(t)\rangle &= 0, \quad j \neq k. \end{aligned}$$

Therefore,

$$\begin{aligned} \langle v_0(t), f(t)\rangle &= \langle v_0(t), v_0(t)\rangle + \sum_{j=0}^{\infty} \langle v_0(t), w_j(t)\rangle = \langle v_0(t), v_0(t)\rangle \\ \langle w_j(t), f(t)\rangle &= \langle w_j(t), v_0(t)\rangle + \sum_{j=0}^{\infty} \langle w_j(t), w_j(t)\rangle = \langle w_j(t), w_j(t)\rangle \end{aligned}$$

Hence,

$$\int_{-\infty}^{\infty} |f(t)|^2\, dt = \langle f(t), f(t)\rangle = \langle v_0(t), v_0(t)\rangle + \sum_{j=0}^{\infty} \langle w_j(t), w_j(t)\rangle. \tag{2}$$

Consider

$$\begin{aligned} \langle v_0(t), v_0(t)\rangle &= \left\langle \sum_{k=-\infty}^{\infty} a_k^{(0)} \phi(t-k), \sum_{l=-\infty}^{\infty} a_l^{(0)} \phi(t-l) \right\rangle \\ &= \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} a_k^{(0)} \overline{a_l^{(0)}} \langle \phi(t-k), \phi(t-l)\rangle \\ &= \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} a_k^{(0)} \overline{a_l^{(0)}} \delta_{kl} = \sum_{k=-\infty}^{\infty} \left|a_k^{(0)}\right|^2. \end{aligned} \tag{3}$$

Similarly,

$$\begin{aligned} \langle w_j(t), w_j(t)\rangle &= \left\langle \sum_{k=-\infty}^{\infty} b_k^{(j)} 2^{j/2} \psi\left(2^j t - k\right), \sum_{l=-\infty}^{\infty} b_l^{(j)} 2^{j/2} \psi\left(2^j t - l\right) \right\rangle \\ &= \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} b_k^{(j)} \overline{b_l^{(j)}} \left\langle 2^{j/2} \psi\left(2^j t - k\right), 2^{j/2} \psi\left(2^j t - l\right) \right\rangle \\ &= \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} b_k^{(j)} \overline{b_l^{(j)}} \delta_{kl} = \sum_{k=-\infty}^{\infty} \left|b_k^{(j)}\right|^2. \end{aligned} \tag{4}$$

From (2), (3) and (4), we have the Parseval's equality

$$\int_{-\infty}^{\infty} |f(t)|^2\, dt = \sum_{k=-\infty}^{\infty} \left|a_k^{(0)}\right|^2 + \sum_{j=0}^{\infty} \sum_{k=-\infty}^{\infty} \left|b_k^{(j)}\right|^2.$$

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Remark:

- Notice that the Parseval's equality is same as the following result that we have proved for inner product spaces: *If $\{\underline{v}_1, \underline{v}_2, \cdots\}$ form orthonormal basis for an inner product space $\mathcal{V}$ and a vector $\underline{v} \in \mathcal{V}$ is written as $\underline{v} = \sum_{k=1}^{\infty} a_k \underline{v}_k$, then*

$$|\underline{v}|^2 = \langle \underline{v}, \underline{v}\rangle = \sum_{k=1}^{\infty} |a_k|^2.$$

Problem 4. For $j \in \mathbb{Z}$, let $\mathcal{V}_j$ be the space of all signals $f(t) \in L^2$ bandlimited within the interval $\left[-2^j\pi, 2^j\pi\right]$. Consider the signal $\phi(t) := \mathrm{sinc}(t) = \frac{\sin(\pi t)}{\pi t}$. Prove the following.

- The nesting, closure, shrinking and scaling properties that we discussed in the class as part of the multiresolution analysis definition.
- $\{\phi(t-k), k \in \mathbb{Z}\}$ is a shift orthogonal basis for $\mathcal{V}_0$.
- $\phi(t) = \phi(2t) + \sum\limits_{k\in\mathbb{Z}} \frac{2(-1)^k}{(2k+1)\pi}\phi(2t-2k-1)$. (Scaling relation)

Solution. Part 1

Nesting property: If $f(t) \in \mathcal{V}_j$, then $f(t)$ is bandlimited within the interval $\left[-2^j\pi, 2^j\pi\right] \implies f(t)$ is bandlimited within the interval $\left[-2^{j+1}\pi, 2^{j+1}\pi\right] \implies f(t) \in \mathcal{V}_{j+1}$. Therefore, $\mathcal{V}_j \subset \mathcal{V}_{j+1}, \quad j = 0, 1, \cdots$. Therefore

$$\mathcal{V}_0 \subset \mathcal{V}_1 \subset \mathcal{V}_2 \subset \mathcal{V}_3 \cdots$$

Scaling property: If $f(t) \in \mathcal{V}_j$, then $f(t)$ is bandlimited within the interval $\left[-2^j\pi, 2^j\pi\right] \implies f(2t)$ is bandlimited within the interval $\left[-2^{j+1}\pi, 2^{j+1}\pi\right] \implies f(2t) \in \mathcal{V}_{j+1}$. Therefore, $\{f(2t) \mid f(t) \in \mathcal{V}_j\} \subseteq \mathcal{V}_{j+1}$.

Similarly, if $f(t) \in \mathcal{V}_{j+1}$, then $f(t)$ is bandlimited within the interval $\left[-2^{j+1}\pi, 2^{j+1}\pi\right] \implies f\left(\frac{t}{2}\right)$ is bandlimited within the interval $\left[-2^j\pi, 2^j\pi\right] \implies f\left(\frac{t}{2}\right) \in \mathcal{V}_j$. Therefore, $\mathcal{V}_j \supset \left\{f\left(\frac{t}{2}\right) \mid f(t) \in \mathcal{V}_{j+1}\right\} \implies \{f(2t) \mid f(t) \in \mathcal{V}_j\} \supseteq \mathcal{V}_{j+1}$.

Therefore, we have $\{f(2t) \mid f(t) \in \mathcal{V}_j\} = \mathcal{V}_{j+1}$.

Shrinking property: Signals in $\mathcal{V}_{-j}$ are bandlimited within the interval $\left[-2^{-j}\pi, 2^{-j}\pi\right]$ for $j \geq 0$. Therefore, signals in $\bigcap\limits_{j\in\mathbb{Z}} \mathcal{V}_j$ will have only a d.c. component in the signal. The only square integrable d.c. signal is $f(t) = 0$. Therefore, $\bigcap\limits_{j\in\mathbb{Z}} \mathcal{V}_j = \{0\}$.

Closure property: Let $f(t) \in L^2(\mathbb{R})$ be any function in L^2 space. Consider its Fourier transform $F(\omega)$. Let $f_j(t)$ be the projection of $f(t)$ onto $\mathcal{V}_j$. Then its Fourier transform is

$$F_j(\omega) = \begin{cases} F(\omega) & \omega \in \left[-2^{-j}\pi, 2^{-j}\pi\right] \\ 0 & \text{otherwise.} \end{cases}$$

For the closure property, we need to show that $\lim\limits_{j\to\infty} f_j(t) = f(t)$ in L^2 sense.

Split the frequencies into disjoint intervals given by $I_j = \left[-2^{-(j+1)}\pi, 2^{-(j+1)}\pi\right] \setminus \left[-2^{-j}\pi, 2^{-j}\pi\right], j = 1, 2, \cdots$ and $I_0 = [-\pi, \pi]$. We have $\bigcup\limits_{j=0}^{N} I_j = \left[-2^{-(N+1)}\pi, 2^{-(N+1)}\pi\right]$ and $\bigcup\limits_{j=0}^{N} I_j = \mathbb{R}$.

Let E be the energy in the signal $f(t)$. Using Parseval's theorem, we have

$$E = \int_{-\infty}^{\infty} |f(t)|^2 \, dt = \int_{-\infty}^{\infty} |F(\omega)|^2 \, d\omega = \sum_{j=0}^{\infty} \int_{\omega\in I_j} |F(\omega)|^2 \, d\omega. \tag{5}$$

Let E_j be the energy of the signal in the frequencies I_j i.e.,

$$E_j = \int_{\omega\in I_j} |F(\omega)|^2 \, d\omega, \quad j = 0, 1, 2, \cdots$$

Then, the equation (5) can be written as

$$\begin{aligned} E &= \sum_{j=0}^{\infty} E_j \\ \implies \lim_{N\to\infty} \left(E - \sum_{j=0}^{N} E_j\right) &= 0. \quad (\because E \text{ is finite}) \end{aligned} \tag{6}$$

Notice that $\sum\limits_{j=0}^{N} E_j$ is the energy of the signal $f(t)$ within the frequencies $\left[-2^{-(N+1)}\pi, 2^{-(N+1)}\pi\right]$. Therefore, $\sum\limits_{j=0}^{N} E_j$ is the energy in $F_{N+1}(\omega)$. Therefore, $\left(E - \sum_{j=0}^{N} E_j\right)$ is the energy in $F(\omega) - F_{N+1}(\omega)$ ($= f(t) - f_{N+1}(t)$ in the time-domain). From equation (6),

$$\lim_{N\to\infty} \int_{-\infty}^{\infty} |f(t) - f_{N+1}(t)|^2 \, dt = 0$$

$$\Longrightarrow \lim_{N\to\infty} f_N(t) = f(t) \quad \left(\text{in } L^2 \text{ sense}\right)$$

Hence, any function in $L^2(\mathbb{R})$ can be represented within $\bigcup_{j=0}^{N} \mathcal{V}_j$. Therefore, $\bigcup_{j=0}^{N} \mathcal{V}_j = L^2(\mathbb{R})$.

Part 2:

We need to prove that

a) $\{\phi(t-k), k \in \mathbb{Z}\}$ are orthogonal and

b) $\{\phi(t-k), k \in \mathbb{Z}\}$ form basis for $\mathcal{V}_0$

To prove that $\{\phi(t-k), k \in \mathbb{Z}\}$ are orthogonal, consider the inner product between $\phi(t-k)$ and $\phi(t-l)$ for $k \neq l$,

$$\begin{aligned}
\int_{-\infty}^{\infty} \phi(t-k)\,\phi(t-l) &= \int_{-\infty}^{\infty} \frac{\sin(\pi(t-k))}{\pi(t-k)} \frac{\sin(\pi(t-l))}{\pi(t-l)} dt \\
&= \int_{-\infty}^{\infty} (-1)^k \frac{\sin(\pi t)}{\pi(t-k)} (-1)^l \frac{\sin(\pi t)}{\pi(t-l)} dt \quad \left(\sin(t-n\pi) = (-1)^n \sin(t)\right) \\
&= \frac{(-1)^{l+k}}{\pi^2(k-l)} \int_{-\infty}^{\infty} \left(\frac{\sin^2(\pi t)}{t-k} - \frac{\sin^2(\pi t)}{t-l}\right) dt \\
&= \frac{(-1)^{l+k}}{\pi^2(k-l)} \int_{-\infty}^{\infty} \frac{\sin^2(\pi(t-k))}{t-k} dt \\
&\quad - \frac{(-1)^{l+k}}{\pi^2(k-l)} \int_{-\infty}^{\infty} \frac{\sin^2(\pi(t-l))}{t-l} dt \quad \left(\sin^2(t-n\pi) = \sin^2(t)\right) \\
&= \frac{(-1)^{l+k}}{\pi^2(k-l)} \int_{-\infty}^{\infty} \frac{\sin^2(\pi t)}{t} dt \quad (\text{Change of variables } t-k \to t) \\
&\quad - \frac{(-1)^{l+k}}{\pi^2(k-l)} \int_{-\infty}^{\infty} \frac{\sin^2(\pi t)}{t} dt \quad (\text{Change of variables } t-l \to t) \\
\int_{-\infty}^{\infty} \phi(t-k)\,\phi(t-l) &= 0
\end{aligned}$$

Proving that $\{\phi(t-k), k \in \mathbb{Z}\}$ form basis for $\mathcal{V}_0$ following from Nyquist sampling theorem. Any bandlimited signal can be sampled at Nyquist rate without loosing any information. The original signal can be constructed from the samples using "sinc-interpolation".

Consider any signal $f(t) \in \mathcal{V}_0$ that is bandlimited to $[-\pi, \pi]$. The Nyquist rate is $2 \times \frac{\pi}{2\pi} = 1$samp/sec. The sampled signal is

$$f_s(t) = \sum_{k\in\mathbb{Z}} f(k)\,\delta(t-k).$$

If $F(\omega)$ is the Fourier transform of $f(t)$, then the Fourier transform of $f_s(t)$ is same as $F(\omega)$ repeated at 2π intervals. Therefore, the signal $f(t)$ can be obtained from $f_s(t)$ by using an ideal low pass filter bandlimited to $[-\pi, \pi]$. In time-domain, this filter is given by $\phi(t) = \frac{\sin(\pi t)}{\pi t}$. Therefore,

$$\begin{aligned}
f(t) &= \sum_{j=-\infty}^{\infty} f_s(t-j)\,\phi(j) \\
&= \sum_{j=-\infty}^{\infty} \sum_{k\in\mathbb{Z}} f(k)\,\delta(t-j-k)\,\phi(j) \\
&= \sum_{k\in\mathbb{Z}} f(k) \sum_{j=-\infty}^{\infty} \delta(t-k-j)\,\phi(j) \\
f(t) &= \sum_{k\in\mathbb{Z}} f(k)\,\phi(t-k) \quad \left(\phi(t) * \delta(t-k) = \phi(t-k)\right)
\end{aligned}$$

Therefore, any function in $\mathcal{V}_0$ can be represented as a linear combination of $\{\phi(t-k), k \in \mathbb{Z}\}$.

Hence, $\{\phi(t-k), k \in \mathbb{Z}\}$ form orthogonal basis for $\mathcal{V}_0$.

Part 3

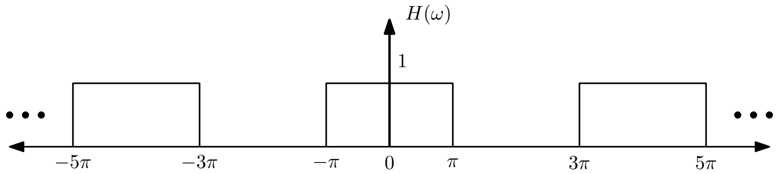

FIGURE 2. Filtering $\phi(2t)$ by this filter $H(\omega)$ results in $\phi(t)$

$\phi(t)$ is an ideal low pass filter band limited to $[-\pi,\pi]$. Its scale $\phi(2t)$ an ideal low pass filter band limited to $[-2\pi,2\pi]$. $\phi(t)$ can be obtained from $\phi(2t)$ by passing through a filter $h(t)$ whose frequency response is as shown in Figure 2. The signal $h(t)$ is given by (derivation for this is given later)

$$h(t) = \delta(t) + \sum_{k\in\mathbb{Z}} \frac{2(-1)^k}{(2k+1)\pi}\delta(t-2k-1). \tag{7}$$

Therefore,

$$\begin{aligned}
\phi(t) &= h(t) * \phi(2t) \\
&= \delta(t) * \phi(2t) + \sum_{k\in\mathbb{Z}} \frac{2(-1)^k}{(2k+1)\pi}\delta(t-2k-1) * \phi(2t) \\
\phi(t) &= \phi(2t) + \sum_{k\in\mathbb{Z}} \frac{2(-1)^k}{(2k+1)\pi}\phi(2t-2k-1)
\end{aligned}$$

Proof for equation (7):

The frequency response in Figure 2, is an even periodic function with period 2π. Therefore, we can write the frequency response as

$$\begin{aligned}
H(\omega) &= a_0 + \sum_{n=1}^{\infty} a_n \cos\left(n\frac{\omega}{2}\right), \\
\text{where, } a_0 &= \int_{-\pi}^{\pi} H(\omega)\, d\omega = 2\pi \\
a_n &= \int_{-\pi}^{\pi} H(\omega)\cos\left(n\frac{\omega}{2}\right) d\omega \\
&= \int_{-\pi}^{\pi} \cos\left(n\frac{\omega}{2}\right) d\omega \\
&= \frac{2}{n}\sin\left(n\frac{\omega}{2}\right)\Big|_{-\pi}^{\pi} \\
&= \frac{4}{n}\sin\left(n\frac{\pi}{2}\right) \\
&= \begin{cases} 0 & n \text{ is even} \\ (-1)^k \frac{4}{2k+1}, & n = 2k+1 \text{ is odd} \end{cases}
\end{aligned}$$

Therefore,

$$\begin{aligned}
H(\omega) &= 2\pi + \sum_{k=0}^{\infty} (-1)^k \frac{4}{2k+1}\cos\left((2k+1)\frac{\omega}{2}\right) \\
&= 2\pi + \sum_{k=0}^{\infty} (-1)^k \frac{2}{2k+1}\left(e^{j(2k+1)\frac{\omega}{2}} + e^{-j(2k+1)\frac{\omega}{2}}\right) \\
\Longrightarrow h(t) &= \delta(t) + \sum_{k=0}^{\infty} (-1)^k \frac{1}{(2k+1)\pi}\left(\delta(t-2k-1) + \delta(t+2k+1)\right) \\
&= \delta(t) + \sum_{k=0}^{\infty} (-1)^k \frac{1}{(2k+1)\pi}\delta(t-2k-1) + \sum_{k=0}^{\infty} (-1)^k \frac{1}{(2k+1)\pi}\delta(t+2k+1)
\end{aligned}$$

$$
\begin{aligned}
&= \delta(t) + \sum_{k=0}^{\infty} (-1)^k \frac{1}{(2k+1)\pi} \delta(t-2k-1) \\
&\quad + \sum_{l=-\infty}^{-1} (-1)^{-1-l} \frac{1}{(2(-l-1)+1)\pi} \delta(t+2(-l-1)+1) \\
&= \delta(t) + \sum_{k=0}^{\infty} (-1)^k \frac{1}{(2k+1)\pi} \delta(t-2k-1) + \sum_{l=-\infty}^{-1} (-1)^{l+1} \frac{2}{-(2l+1)\pi} \delta(t-2l-1) \\
&= \delta(t) + \sum_{k=0}^{\infty} (-1)^k \frac{1}{(2k+1)\pi} \delta(t-2k-1) + \sum_{l=-\infty}^{-1} (-1)^{l} \frac{2}{(2l+1)\pi} \delta(t-2l-1) \\
&= \delta(t) + \sum_{k=-\infty}^{\infty} (-1)^k \frac{1}{(2k+1)\pi} \delta(t-2k-1)
\end{aligned}
$$

■