

Solutions Key

Indian Institute of Science

E9-252: Mathematical Methods and Techniques in Signal Processing

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Mid Term Exam#1, Fall 2015

Name and SR.No:

Instructions:

- Only four A4 sheets of paper with written notes on both sides are allowed .
- The time duration is 3 hrs.
- There are five main questions. None of them have negative marking.
- Attempt all of them with careful reasoning and justification for partial credit.
- Do not panic, do not cheat.
- Good luck!

Question No.	Points scored
1	
2	
3	
4	
5	
Total points	

PROBLEM 1: Examine if the following statements are true or false with correct reasoning. Random guessing or incorrect reasoning fetches zero credit. A statement is true if it is generic for all cases. A counter example is enough to make it false. All sub-parts of this problem carry equal credit.

- (1) Let \bar{v} be the eigenvector for a square matrix A with a corresponding eigenvalue zero. \bar{v} is also an eigenvector for A^n for integers $n > 1$.
- (2) Consider the random process $s(t) = A \sin(\omega t)$, where A is Gaussian distributed with mean zero and variance σ^2 . The frequency ω is deterministic. $s(t)$ is a stationary process.
- (3) Upsampling of a non-zero discrete time signal by a factor $L > 1$ always causes images within the base band.
- (4) Consider a discrete time signal $x[n]$ at D samples/s coded at c bits/sample with energy predominantly in the low pass region. Suppose we pass this signal through a two channel QMF bank so that the output of the analysis bank is coded at a bits and b bits per sample in the low and high frequency subbands respectively during transmission. We can achieve a compression in the data rate if $2c > a + b$.
- (5) If an LTI system is causal, it is always stable.

(20 pts.)

1) $A\bar{v} = 0 \cdot \bar{v}$ ($\lambda = 0$) From eigenvalue eqn

Try $A^2\bar{v} = A \cdot A\bar{v} = A \cdot \bar{0} = 0$

Continuing, $A^n\bar{v} = \bar{0} \Rightarrow \bar{v}$ is also an eigenvector for A^n $n > 1$.
 ($\because A^{n-1} A\bar{v} = 0 \cdot \bar{v}$) TRUE

2) Consider the 3rd moment and higher... (odd ones = 0, even $\neq 0$)
 $E(s^4(t)) = E(A^4) \sin^4(\omega t)$
 $E(A^4) \neq 0$ for Gaussian \Rightarrow 4th order statistics is a function of 't'
 \Rightarrow Not strict sense stationary. FALSE

3) DFT is periodic with $2\pi \Rightarrow$ \uparrow by L causes those spectra beyond base band to be wrapped in due to compression
 \therefore Statement is True

4) Bit rate originally is Dc bits/s
 At the subbands, bit rate = $\frac{D}{2}a + \frac{D}{2}b$
 from 2-channel QMF due to downsampling by 2
 \Rightarrow Compression rate = $\frac{Dc}{\frac{D}{2}(a+b)}$ ($CR > 1 \Rightarrow 2c > a+b$)
TRUE

5) Suppose $h[n] = u[n]$ (Causal)
 $\sum_{n=-\infty}^{\infty} h(n) = \infty$ (Not stable) FALSE

PROBLEM 2: This problem has 3 parts.

- (1) Let $\{\bar{v}_i\}_{i=1}^n$ be a collection of orthonormal vectors and $\{a_i\}_{i=1}^n$ be scalars.
 Prove that $\|\sum_{i=1}^n a_i \bar{v}_i\|^2 = \sum_{i=1}^n |a_i|^2$. (8 pts.)
- (2) Let the vector space $\mathcal{V}_1 \perp \mathcal{V}_2$.
 (a) Prove that \mathcal{V}_2 is a subspace of \mathcal{V}_1^\perp . State the converse and verify if it is true.
 (b) Prove that $\mathcal{V}_1 \cap \mathcal{V}_2 = \{\bar{0}\}$. (8 pts.)
- (3) Consider the signals $f_1(t) = 1$ and $f_2(t) = t$ over $[-1, 1]$. Are they orthonormal? Obtain a closest linear signal approximation to $s(t) = t^2$ over $[-1, 1]$. Plot the approximate signal representation on a signal coordinate system. (9 pts.)

1) Consider $\left(\sum_{i=1}^n a_i \bar{v}_i \right) \left(\sum_{i=1}^n a_i \bar{v}_i \right)^\dagger$

$$= \sum_{i=1}^n a_i a_i^* \bar{v}_i \bar{v}_i^\dagger \quad \left(\begin{array}{l} \because \bar{v}_i \bar{v}_i^\dagger = 0 \\ \bar{v}_i \bar{v}_i^\dagger = 1 \end{array} \right)$$

$$= \sum_{i=1}^n |a_i|^2 \quad \square$$

(2) (a) Since $\mathcal{V}_1 \perp \mathcal{V}_2 \Rightarrow$ every vector in \mathcal{V}_2 is orthogonal to all vectors in \mathcal{V}_1 .
 \mathcal{E}_1 lies within \mathcal{V}_1^\perp by definition of \mathcal{V}_1^\perp .
 $\Rightarrow \mathcal{V}_2 \subset \mathcal{V}_1^\perp$

CONVERSE If $\mathcal{V}_2 \subset \mathcal{V}_1^\perp$, then $\mathcal{V}_1 \perp \mathcal{V}_2$.
 If $\mathcal{V}_2 \subset \mathcal{V}_1^\perp$, every vector in \mathcal{V}_2 is in \mathcal{V}_1^\perp .
 \Rightarrow Every vector in \mathcal{V}_2 is orthogonal to every vector in \mathcal{V}_1 .
 $\Rightarrow \mathcal{V}_1 \perp \mathcal{V}_2 \quad \square$

(b) Since $\bar{0}$ is part of every vector space, $\mathcal{V}_1 \cap \mathcal{V}_2 = \bar{0}$. But, we must prove that there is no other vector \bar{v} other than $\bar{0}$ in $\mathcal{V}_1 \cap \mathcal{V}_2$. Let us prove by contradiction.
 Suppose $\bar{v} \in \mathcal{V}_1 \ \& \ \bar{v} \in \mathcal{V}_2$ since $\mathcal{V}_1 \perp \mathcal{V}_2$
 $\Rightarrow \bar{v} \perp \bar{v} \Rightarrow \langle \bar{v}, \bar{v} \rangle = 0 \Rightarrow \bar{v}$ must be $\bar{0}$ or none else. \square

3) Consider $f_1(t) = 1$ $f_2(t) = t$ over $[-1, 1]$
 $\int_{-1}^1 f_1(t) f_2(t) dt = 0$ (Orthogonal)

However $\|f_2(t)\| = \left(\int_{-1}^1 t^2 dt \right)^{\frac{1}{2}} = \sqrt{\frac{2}{3}} \neq 1$

\therefore They are not orthonormal.

Let us form an orthonormal basis

$$\phi_1(t) = \frac{f_1(t)}{\|f_1(t)\|} = \frac{1}{\sqrt{2}} \quad -1 \leq t \leq 1$$

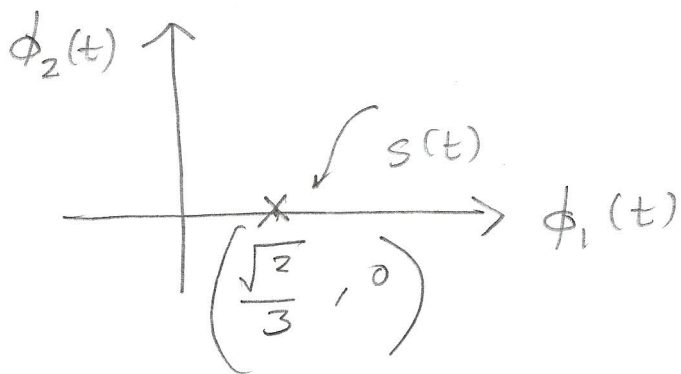
$$\phi_2(t) = \frac{f_2(t)}{\|f_2(t)\|} = \frac{f_2(t)}{(\sqrt{2/3})} \quad -1 \leq t \leq 1$$

To get the coordinates of $s(t)$ in $\phi_1(t) - \phi_2(t)$ plane, we need

$$\langle s(t), \phi_1(t) \rangle \quad \text{and} \quad \langle s(t), \phi_2(t) \rangle$$

$$\langle s(t), \phi_1(t) \rangle = \frac{1}{\sqrt{2}} \int_{-1}^1 t^2 dt = \frac{\sqrt{2}}{3}$$

$$\langle s(t), \phi_2(t) \rangle = \sqrt{\frac{3}{2}} \int_{-1}^1 t^3 dt = 0$$



(Integral of an odd function)

PROBLEM 3: Consider the structure shown in Figure 1, where W is the 3×3 DFT matrix. This is a three channel synthesis bank with three filters $F_0(z)$, $F_1(z)$ and $F_2(z)$. (For example, $F_0(z) = Y(z)/Y_0(z)$ with $y_1[n]$ and $y_2[n]$ set to zero.)

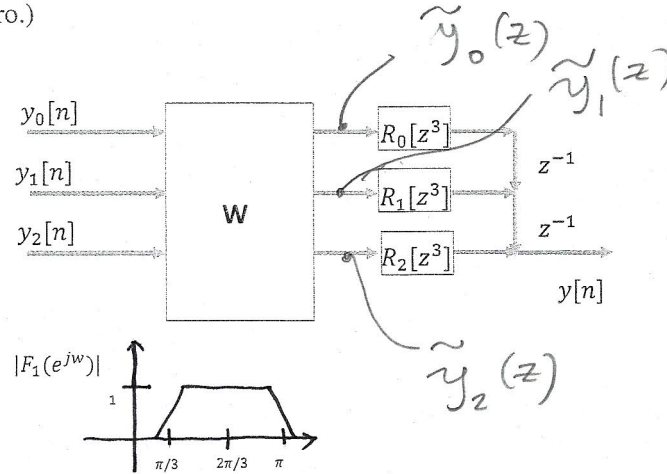


FIGURE 1. Three channel synthesis bank

- (1) Assuming $R_0(z) = 1 + z^{-1}$, $R_1(z) = 1 - z^{-2}$, $R_2(z) = 2 + 3z^{-1}$, find an expression for the three synthesis filters $F_0(z)$, $F_1(z)$ and $F_2(z)$. (15 pts.)
- (2) Let the magnitude of $F_1(z)$ be as shown above. Plot the frequency response of $|F_0(e^{j\omega})|$ and $|F_2(e^{j\omega})|$. (5 pts.)

$$1) Y(z) = \begin{bmatrix} z^{-2} R_0(z^3) & z^{-1} R_1(z^3) & R_2(z^3) \end{bmatrix} \begin{bmatrix} \tilde{y}_0(z) \\ \tilde{y}_1(z) \\ \tilde{y}_2(z) \end{bmatrix}$$

$$\text{But } \begin{bmatrix} \tilde{y}_0(z) \\ \tilde{y}_1(z) \\ \tilde{y}_2(z) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega^4 \end{bmatrix} \begin{bmatrix} y_0(z) \\ y_1(z) \\ y_2(z) \end{bmatrix}$$

$$\text{Where } \omega = e^{-j2\pi/3} \quad \omega^4 = \omega$$

Plug in $y_1(z)$ and $y_2(z) = 0$ to get $F_0(z)$

$$\begin{aligned} \therefore F_0(z) &= z^{-2} R_0(z^3) + z^{-1} R_1(z^3) + R_2(z^3) \\ &= z^{-2} (1 + z^{-3}) + z^{-1} (1 - z^{-6}) + 2 + 3z^{-1} \\ &= 2 + z^{-1} + z^{-2} + 3z^{-3} + z^{-5} - z^{-7} \end{aligned}$$

In a similar way setting the other points zero

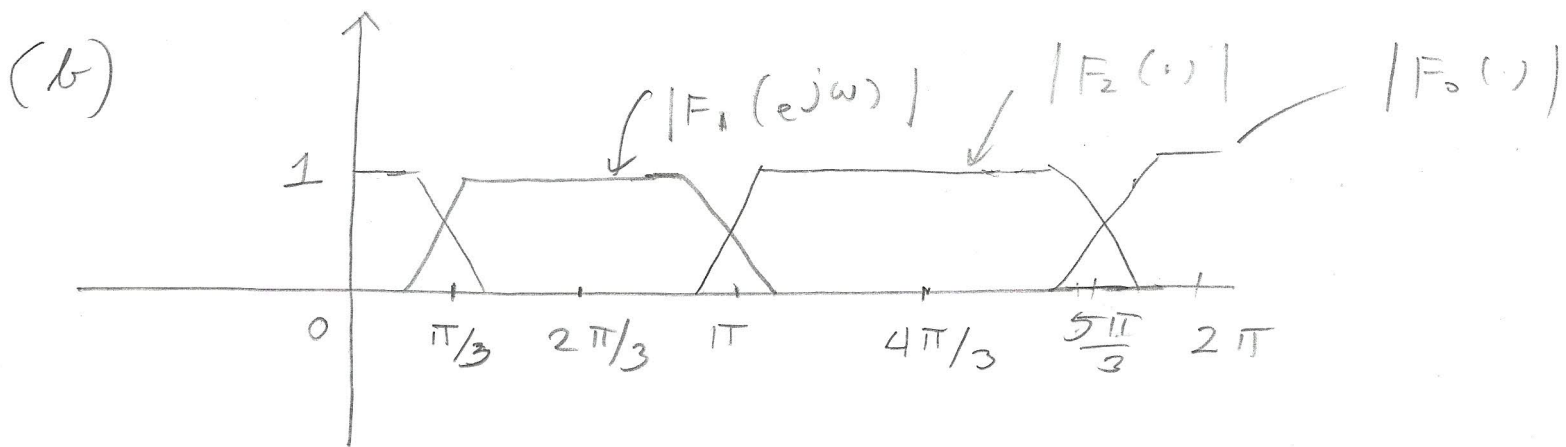
$$F_1(z) = 2\omega^2 + \omega z^{-1} + z^{-2} + 3\omega^2 z^{-3} + z^{-5} - \omega z^{-7}$$

$$F_2(z) = 2\omega + \omega^2 z^{-1} + z^{-2} + 3\omega z^{-3} + z^{-5} - \omega^2 z^{-7}$$

$$F_1(z) = \omega^2 F_0(\omega z)$$

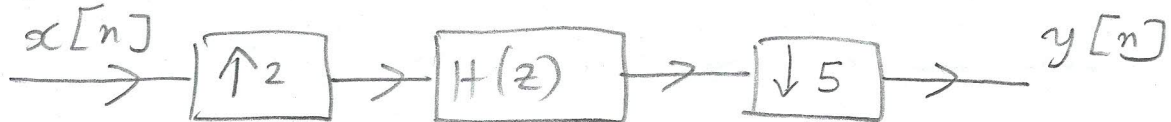
$$F_2(z) = \omega F_0(\omega^2 z)$$

These are useful for the next part for spectra sketch.

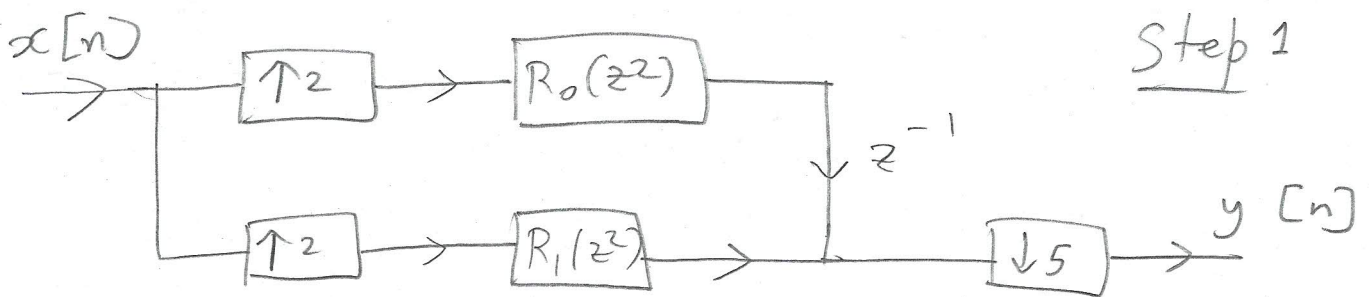


PROBLEM 4: A certain sampling rate conversion system requires downsampling a signal at 100 Msamples/s to 40 Msamples/s. From first principles, derive a fully efficient architecture using downsamplers and expanders. Sketch the schematic of your multirate system. (25 pts.)

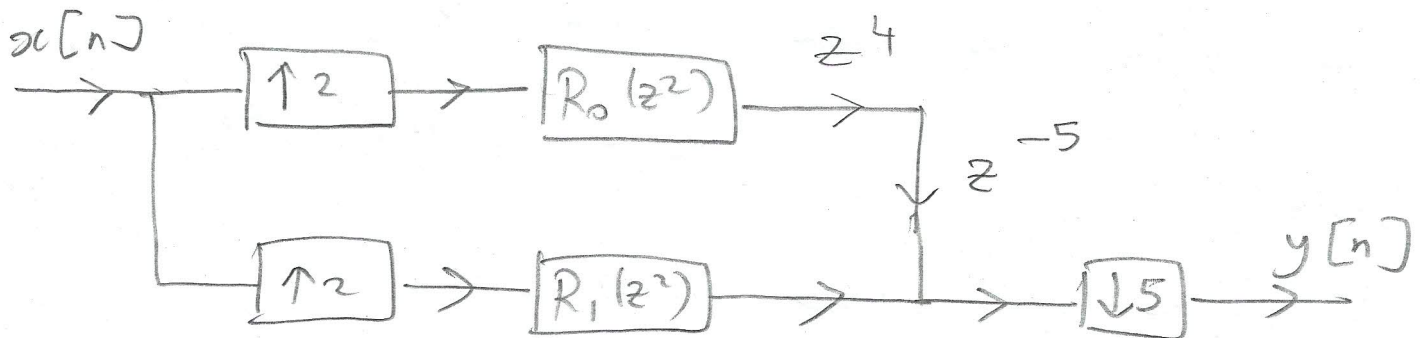
We need a 5:2 reduction in rate. $\frac{100}{40} = \frac{5}{2}$
 So, the sampling rate conversion system with a filter is



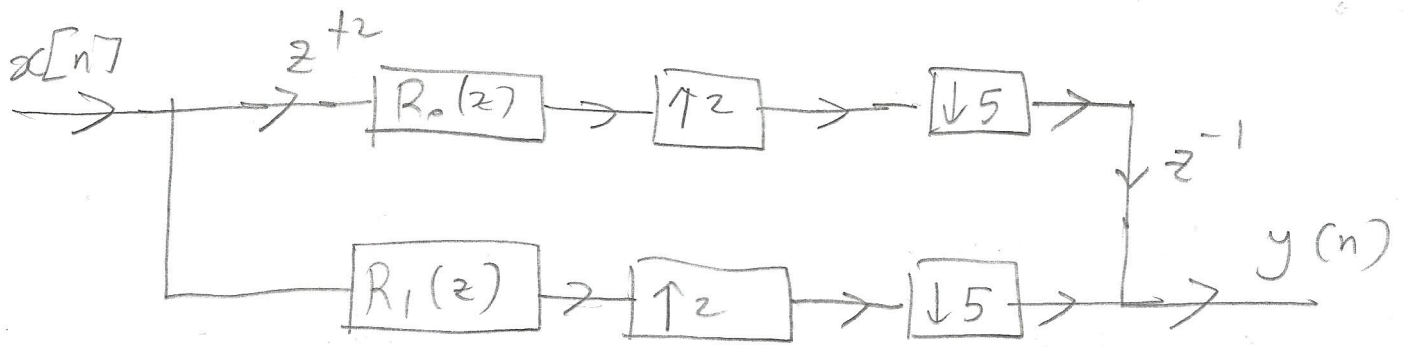
We need to a polyphase architecture similar to Hsiao's work we discussed in class.



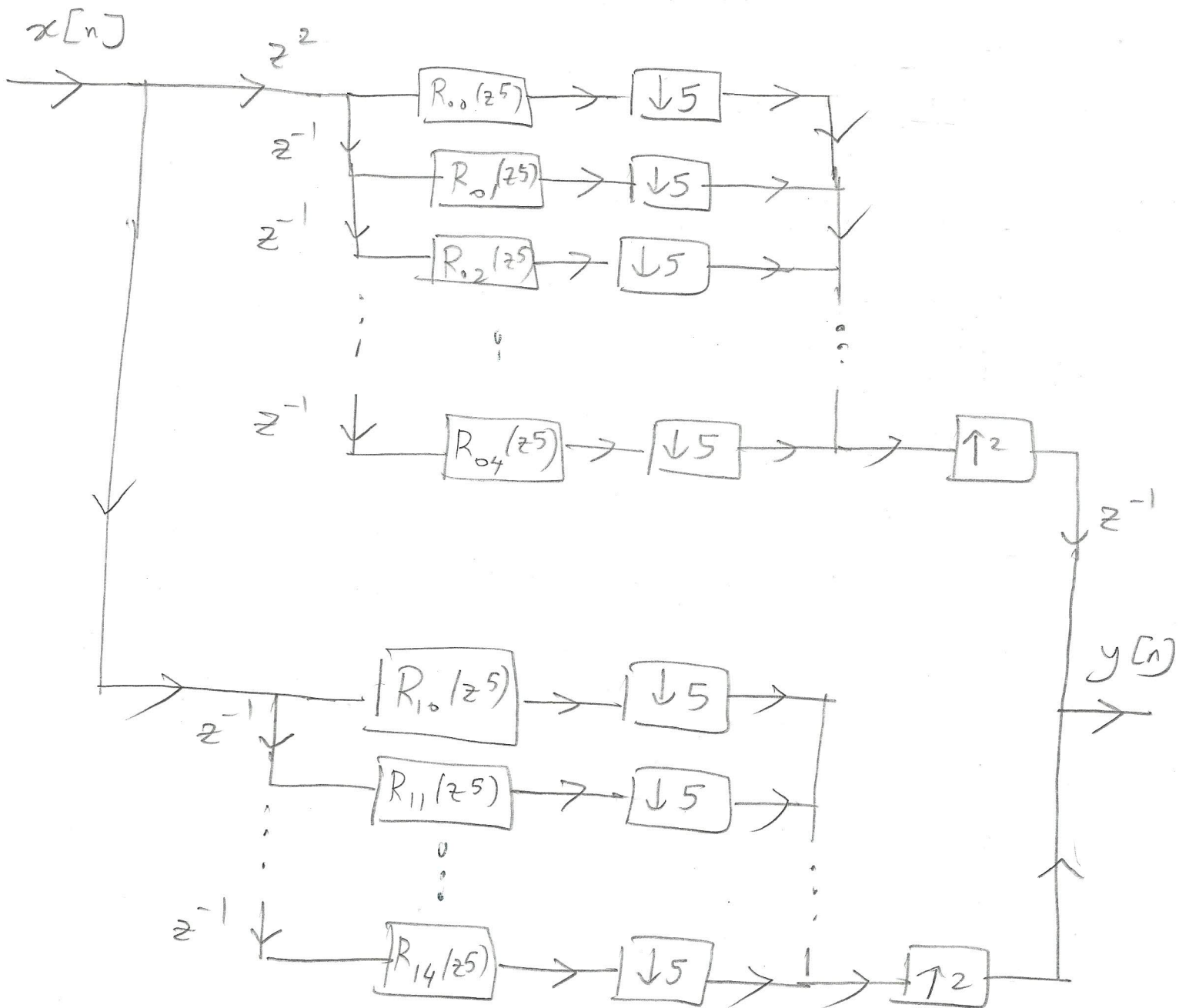
Realize $z^{-1} = z^{-5} \cdot z^4$



Now apply noble identities & fold $\downarrow 5$ into the feed forward paths



$\text{gcd}(2, 5) = 1$, We can swap $\uparrow 2$ & $\downarrow 5$



Final architecture

PROBLEM 5: The input to an LTI system is a WSS random process. Is the output WSS? Justify. You can assume that the LTI filter response is absolutely summable. (10 pts.)

$$y(t) = x(t) * h(t) \quad (\because \text{LTI})$$

$$E(y(t)) = E\left(\int_{-\infty}^{\infty} x(\tau) h(t-\tau) d\tau\right)$$

We can swap $E(\cdot)$ and $\int(\cdot)$ due to absolutely summable & consequence of Tonelli's Conditions.

$$\begin{aligned} \therefore E(y(t)) &= \int_{-\infty}^{\infty} E(x(\tau) h(t-\tau)) d\tau \\ &= \mu_x \int_{-\infty}^{\infty} h(t-\tau) d\tau = \mu_x H(0) \end{aligned}$$

dc term of Cent. Fourier tr.

Consider

$$E(y(t_1) y(t_2))$$

From our class notes, this is just.

$$R_{yy}(t_1, t_2) = R_{xx}(t_2 - t_1) * h(t_2 - t_1) * h^*(t_2 - t_1)$$

\Rightarrow It is a WSS process