E9-252:Mathematical methods and techniques in signal processing
HW 4 solutions

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Problem 1

(1)

Without loss of generality we shift the signal to make it causal

$$f(t) = \begin{cases} 3 & -2 \leq t < -1 \\ -4 & -1 \leq t < 0 \\ 2 & 0 \leq t < 1 \\ 1 & 1 \leq t < 2 \end{cases} \tag{1}$$

After shift

$$f(t) = \begin{cases} 3 & 0 \leq t < 1 \\ -4 & 1 \leq t < 2 \\ 2 & 2 \leq t < 3 \\ 1 & 3 \leq t < 4 \end{cases} \tag{2}$$

We project the signal $f(t) = f_0(t)$ onto components lying in V_1 and W_1 because $V_0 = V_1 \oplus W_1$.

$$f(t) = \sum_n c_n^{(1)} \Phi_{1,n}(t) + \sum_n d_n^{(1)} \Psi_{1,n}(t)$$

where,

$$\Phi_{j,n}(t) = 2^{-\frac{j}{2}} \Phi(2^{-j}\, t - n)$$

The components $c_n^{(1)}$ and $d_n^{(1)}$ are obtained by

$$c_n^{(1)} = \langle f_0(t), \Phi_{1,n}(t) \rangle$$
$$d_n^{(1)} = \langle f_0(t), \Psi_{1,n}(t) \rangle$$

Writing $f_0(t) = f_1(t) + g_1(t)$ where,
$f_1(t) = \sum_n c_n^{(1)} \Phi_{1,n}(t)$ and $g_1(t) = \sum_n d_n^{(1)} \Psi_{1,n}(t)$. But we can further project the signal $f_1(t)$ lying in V_1 onto components lying in V_2 and W_2 because $V_1 = V_2 \oplus W_2$. Now,

$$f_1(t) = \sum_n c_n^{(2)} \Phi_{2,n}(t) + \sum_n d_n^{(2)} \Psi_{2,n}(t)$$

where,

$$c_n^{(2)} = \langle f_1(t), \Phi_{2,n}(t)\rangle$$
$$d_n^{(2)} = \langle f_1(t), \Psi_{2,n}(t)\rangle$$

Therefore the final wavelet decomposition of the signal is

$$f(t) = \underbrace{\sum_n d_n^{(1)}\Psi_{1,n}(t)}_{\text{I}} + \underbrace{\sum_n c_n^{(2)}\Phi_{2,n}(t)}_{\text{II}} + \underbrace{\sum_n d_n^{(2)}\Psi_{2,n}(t)}_{\text{III}} \tag{3}$$

The coefficients are found to be

$$c_0^{(1)} = -\frac{1}{\sqrt{2}} \quad c_1^{(1)} = \frac{3}{\sqrt{2}} \quad d_0^{(1)} = \frac{7}{\sqrt{2}} \quad d_1^{(1)} = \frac{1}{\sqrt{2}} \Rightarrow$$

$$f_1(t) = \begin{cases} -0.5 & 0 \le t < 2 \\ 1.5 & 2 \le t < 4 \end{cases} \tag{4}$$

$$g_1(t) = \begin{cases} 3.5 & 0 \le t < 1 \\ -3.5 & 1 \le t < 2 \\ 0.5 & 2 \le t < 3 \\ -0.5 & 3 \le t < 4 \end{cases} \tag{5}$$

Writing $f_1(t) = f_2(t) + g_2(t)$ where,
$f_2(t) = \sum_n c_n^{(2)}\Phi_{2,n}(t)$ and $g_2(t) = \sum_n d_n^{(2)}\Psi_{2,n}(t)$.

The coefficients are found to be

$$c_0^{(2)} = 1 \quad d_0^{(2)} = -2 \Rightarrow$$

$$f_2(t) = \begin{cases} 0.5 & 0 \le t < 4 \end{cases} \tag{6}$$

$$g_2(t) = \begin{cases} -1 & 0 \le t < 2 \\ 1 & 2 \le t < 4 \end{cases} \tag{7}$$

(2)

Frequency response can be plotted for each of the signals namely $f(t), f_1(t), g_1(t), f_2(t), g_2(t)$.

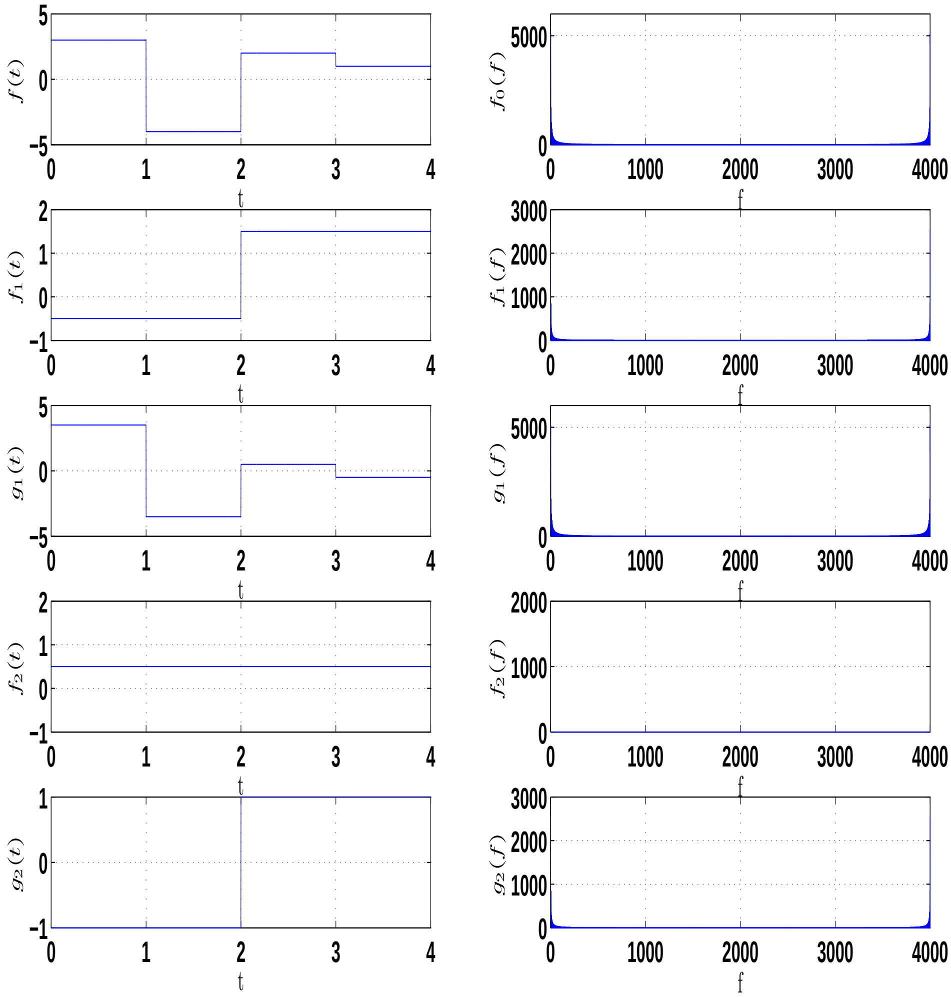

Figure 1: Signal decomposition and Frequency response

(3)

We go from coarser to finer representation of the signal. Terms II and III in (3) represent a coarse representation of the signal. To make the representation finer we add additional information captured in the wavelets $\Psi_{j,n}(t)$. Nulling the last stage would mean equating the coefficient of $\Psi_{2,n}(t)$ to zero. Therefore, the approximate signal is

$$\begin{aligned}\widehat{f}(t) &= \sum_n d_n^{(1)}\Psi_{1,n}(t) + \sum_n c_n^{(2)}\Phi_{2,n}(t) \\ &= \frac{7}{\sqrt{2}}\Psi_{1,0}(t) + \frac{1}{\sqrt{2}}\Psi_{1,1}(t) + \Phi_{2,0}(t) \\ &= \begin{cases} 4 & 0 \le t < 1 \\ -3 & 1 \le t < 2 \\ 1 & 2 \le t < 3 \\ 0 & 3 \le t < 4. \end{cases}\end{aligned}$$

Energy lost in the approximation is

$$E_{\text{loss}} = ||f(t) - \widehat{f}(t)||^2 = 4.$$

$$\text{Fraction lost} = \frac{||f(t) - \widehat{f}(t)||^2}{||f(t)||^2} = \frac{4}{30}.$$

(4)

After performing the wavelet decomposition, we get the wavelet coefficients $d_n^{(1)}, c_n^{(2)}, d_n^{(2)}$. We perform quantization on these coefficients. The quantization is done as follows

$$\Delta = \frac{\max_n\{d_n^{(1)}, c_n^{(2)}, d_n^{(2)}\} - \min_n\{d_n^{(1)}, c_n^{(2)}, d_n^{(2)}\}}{2^3},$$

$$\Delta = \frac{\frac{7}{\sqrt{2}} + 2}{8} = \frac{7\sqrt{2} + 4}{16} = 0.868.$$

Quantization level	binary
max=4.95	111
4.076	110
3.208	101
2.34	100
1.472	011
0.604	010
-1.132	001
min=-2	000

We get the new wavelet coefficients namely $\widehat{c}_n^{(j)}, \widehat{d}_n^{(j)}$

$$\widehat{d}_0^{(1)} = 4.95 \quad \widehat{d}_1^{(1)} = 0.604 \quad \widehat{c}_0^{(2)} = 1.472 \quad \widehat{d}_0^{(2)} = -2.$$

The binary representation is therefore
binary($\widehat{d}_0^{(1)}$)= 111, binary($\widehat{d}_1^{(1)}$)= 010,binary($\widehat{c}_0^{(2)}$)= 011, binary($\widehat{d}_0^{(2)}$)= 000.
The reconstructed signal becomes

$$\begin{aligned}\widehat{f}_q(t) &= \sum_n \widehat{d}_n^{(1)}\Psi_{1,n}(t) + \sum_n \widehat{c}_n^{(2)}\Phi_{2,n}(t) + \sum_n \widehat{d}_n^{(2)}\Psi_{2,n}(t) \\ &= 4.95\Psi_{1,0}(t) + 0.604\Psi_{1,1}(t) + 1.472\Phi_{2,0}(t) - 2\Psi_{2,0}(t)\end{aligned}$$

$$E_{\text{qloss}} = ||f(t) - \widehat{f}_q(t)||^2$$

$$f(t) - \widehat{f}_q(t) = 0.103\Psi_{1,1}(t) - 0.472\Phi_{2,0}(t)$$

$$\text{Q-Fraction lost} = \frac{||f(t) - \widehat{f}_q(t)||^2}{||f(t)||^2} = \frac{0.232}{30} = 0.7\%$$

Note: The translation(time shift) that was done earlier to the signal does not change the quantization and therefore, the quantized signal is time shifted accordingly.

Problem 2

(1)

Calculate the mean vector $\boldsymbol{\mu} = \sum_{i=0}^{6} p_i \boldsymbol{x_i}$ which gives us $\boldsymbol{\mu} = \begin{pmatrix} 0.1875 \\ 0.4375 \end{pmatrix}$.

Calculate the covariance matrix $\boldsymbol{C} = \sum_{i=0}^{6} p_i (\boldsymbol{x_i} - \boldsymbol{\mu})(\boldsymbol{x_i} - \boldsymbol{\mu})^{\boldsymbol{T}}$ which gives us $\boldsymbol{C} = \begin{pmatrix} 4.4023 & 3.5430 \\ 3.5430 & 4.1211 \end{pmatrix}$.

Lets calculate the eigen vectors of the matrix $\boldsymbol{C}$

We find the eigen vectors are $\begin{pmatrix} -0.7210 \\ -0.6929 \end{pmatrix}$ and $\begin{pmatrix} 0.6929 \\ -0.7210 \end{pmatrix}$ with corresponding eigen values $\lambda_1 = 7.8075$ and $\lambda_2 = 0.7160$ respectively. Form the matrix $\boldsymbol{\Phi}$ by stacking as columns, the eigen vectors of $\boldsymbol{C}$

$$\boldsymbol{\Phi} = \begin{pmatrix} -0.7210 & 0.6929 \\ -0.6929 & -0.7210 \end{pmatrix}.$$

The KL representation becomes $\tilde{\boldsymbol{x}}_{\boldsymbol{i}} = \boldsymbol{\Phi}^{\boldsymbol{T}} \boldsymbol{x_i}$ giving us the vectors

$$\tilde{\boldsymbol{x}}_{\boldsymbol{1}} = \begin{pmatrix} 0.0280 \\ -1.4139 \end{pmatrix} \tilde{\boldsymbol{x}}_{\boldsymbol{2}} = \begin{pmatrix} -0.6929 \\ -0.7210 \end{pmatrix} \tilde{\boldsymbol{x}}_{\boldsymbol{3}} = \begin{pmatrix} 4.2418 \\ 0.0841 \end{pmatrix}$$
$$\tilde{\boldsymbol{x}}_{\boldsymbol{4}} = \begin{pmatrix} -0.7210 \\ 0.6929 \end{pmatrix} \tilde{\boldsymbol{x}}_{\boldsymbol{5}} = \begin{pmatrix} -0.0280 \\ 1.4139 \end{pmatrix} \tilde{\boldsymbol{x}}_{\boldsymbol{6}} = \begin{pmatrix} -4.2418 \\ -0.0841 \end{pmatrix}.$$

(2)

New representation keeping only the dominant eigen component is done by taking inner product of the dominant eigen vector namely $\widehat{\boldsymbol{e}} = \begin{pmatrix} -0.7210 \\ -0.6929 \end{pmatrix}$

$$\widehat{\boldsymbol{x_i}} = \langle \boldsymbol{x_i}, \widehat{\boldsymbol{e}} \rangle \, \widehat{\boldsymbol{e}}$$

This gives us

$$\widehat{\boldsymbol{x}}_{\boldsymbol{1}} = \begin{pmatrix} -0.202 \\ -0.0194 \end{pmatrix} \widehat{\boldsymbol{x}}_{\boldsymbol{2}} = \begin{pmatrix} 0.4996 \\ 0.4802 \end{pmatrix} \widehat{\boldsymbol{x}}_{\boldsymbol{3}} = \begin{pmatrix} -3.0583 \\ -2.9393 \end{pmatrix}$$
$$\widehat{\boldsymbol{x}}_{\boldsymbol{4}} = \begin{pmatrix} 0.5198 \\ 0.4996 \end{pmatrix} \widehat{\boldsymbol{x}}_{\boldsymbol{5}} = \begin{pmatrix} 0.202 \\ 0.0194 \end{pmatrix} \widehat{\boldsymbol{x}}_{\boldsymbol{6}} = \begin{pmatrix} 3.0583 \\ 2.9393 \end{pmatrix}.$$

All points point in the direction of $\widehat{\boldsymbol{e}}$.

All the points lie on the straight line $y = 0.9610x$.

Fraction of the signal energy lost is $\frac{\lambda_2}{\lambda_1 + \lambda_2} = 8.4\%$

(3)

Let the equation of the straight line that minimizes the squared distance from the original points be $y = mx + c$. We need to evaluate the optimal m and c. The cost function that needs to be minimized is $\mathcal{E} = \sum_{i=0}^{6} p_i(mx_i + c - y_i)^2$, where x_i and y_i refer to the first and second components of $\boldsymbol{x_i}$ respectively.

$$\frac{\partial \mathcal{E}}{\partial m} = 0 \quad \& \quad \frac{\partial \mathcal{E}}{\partial c} = 0 \Rightarrow$$

$$71m + 3c = 58 \quad \& \quad 3m + 16c = 7 \Rightarrow$$

$$m = 0.8048 \quad \& \quad c = 0.2866$$

(4)

After dimensionality reduction, the points lie on a straight line $y = 0.9610x$. The optimal line that separates the reduced points must be perpendicular to the straight line $y = 0.9610x$. Let the equation of the straight line that minimizes the squared distance from the points after dimensionality reduction be $0.9610y + x = c$. We need to evaluate the optimal c. The cost function that needs to be minimized is $\widehat{\mathcal{E}} = \sum_{i=0}^{6} p_i(0.9610y_i + x_i - c)^2$, where x_i and y_i refer to the first and second components of $\widehat{\boldsymbol{x_i}}$ respectively.

$$\frac{\partial \widehat{\mathcal{E}}}{\partial c} = 0 \Rightarrow$$

$$16c = 11.7660 \Rightarrow c = 0.7354$$