INDIAN INSTITUTE OF SCIENCE
E9-252: MATHEMATICAL METHODS AND TECHNIQUES IN SIGNAL PROCESSING
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Problem 1. (5.16) We use following properties of determinant under elementary row operations:
(a) When a row of a matrix is scaled by a scalar factor, its determinant is also scaled by the same factor.
(b) When two rows of a matrix are exchanged, the sign of its determinant changes.
By scaling the last r rows of the matrix

$$\mathbf{A}=\begin{bmatrix}\mathbf{0} & \mathbf{I}_{M-r}\\ z^{-1}\mathbf{I}_r & \mathbf{0}\end{bmatrix}$$

by the factor z, we have

$$\begin{aligned} z^r\det(\mathbf{A}) &= \det\left(\begin{bmatrix}\mathbf{0} & \mathbf{I}_{M-r}\\ \mathbf{I}_r & \mathbf{0}\end{bmatrix}\right)\\ \implies \det(\mathbf{A}) &= z^{-r}\det\left(\begin{bmatrix}\mathbf{0} & \mathbf{I}_{M-r}\\ \mathbf{I}_r & \mathbf{0}\end{bmatrix}\right).\end{aligned}$$

Note that the the matrix

$$\mathbf{B}=\begin{bmatrix}\mathbf{0} & \mathbf{I}_{M-r}\\ \mathbf{I}_r & \mathbf{0}\end{bmatrix}$$

can be transformed to $\mathbf{I}_M$ by using the row exchange operations. Therefore $\det(\mathbf{B})$ is of the form $\pm\det(\mathbf{I}_M)=\pm 1$.
Hence, $\det(\mathbf{A})$ is of the form $\pm z^{-r}$.
Remark:
The matrix $\mathbf{B}$ can be transformed to $\mathbf{I}_M$ using $r(M-r)$ row exchange operations using following procedure (bubble sort algorithm). For $i=1$ to r, shift the $(M-r+i)^{th}$ row to the top by $(M-r)$ rows bringing down the rows $\begin{bmatrix}\mathbf{0} & \mathbf{I}_{M-r}\end{bmatrix}$ by 1 position. This involves $(M-r)$ row exchanges as given by $\mathbf{R}_{M-r+i-j+1}\leftrightarrow\mathbf{R}_{M-r+i-j}$ for $j=1$ to $(M-r)$. Note that $r(M-r)$ is not the least number of row exchange operations.
Therefore, $\det(\mathbf{A})=(-1)^{r(M-r)}z^{-r}$.

Problem 2. (**5.20**) Define

$$\begin{aligned}\mathbf{E}(z) &= \begin{bmatrix}E_0(z) & E_1(z) & \cdots & E_{M-1}(z)\end{bmatrix}^T,\\ \mathbf{H}(z) &= \begin{bmatrix}H_0(z) & H_1(z) & \cdots & H_{M-1}(z)\end{bmatrix}^T,\\ \mathbf{\Lambda}(z) &= \begin{bmatrix}1 & 0 & 0 & & 0\\ 0 & z^{-1} & 0 & \cdots & 0\\ 0 & 0 & z^{-2} & \cdots & \vdots\\ \vdots & \vdots & \vdots & \ddots & 0\\ 0 & 0 & \cdots & 0 & z^{-(M-1)}\end{bmatrix}.\end{aligned}$$

We have

$$\mathbf{H}(z)=\mathbf{W}\mathbf{\Lambda}(z)\mathbf{E}(z^M).$$

(**Part a**)

$$\sum_{k=0}^{M-1}\left|H_k\left(e^{j\omega}\right)\right|^2 = \left(\mathbf{H}\left(e^{j\omega}\right)\right)^\dagger\mathbf{H}\left(e^{j\omega}\right)$$

$$
\begin{aligned}
&= \left(\mathbf{E}(e^{jM\omega})\right)^{\dagger}\left(\mathbf{\Lambda}\left(e^{j\omega}\right)\right)^{\dagger}\mathbf{W}^{\dagger}\mathbf{W}\mathbf{\Lambda}\left(e^{j\omega}\right)\mathbf{E}(e^{jM\omega}) \\
&= M\left(\mathbf{E}(e^{jM\omega})\right)^{\dagger}\left(\mathbf{\Lambda}\left(e^{j\omega}\right)\right)^{\dagger}\mathbf{\Lambda}\left(e^{j\omega}\right)\mathbf{E}(e^{jM\omega}) \quad (\mathbf{W}^{\dagger}\mathbf{W} = M\mathbf{I}) \\
&= M\left(\mathbf{E}(e^{jM\omega})\right)^{\dagger}\mathbf{E}(e^{jM\omega}) \quad \left(\left(\mathbf{\Lambda}\left(\mathbf{e}^{\mathbf{j}\omega}\right)\right)^{\dagger}\mathbf{\Lambda}\left(\mathbf{e}^{\mathbf{j}\omega}\right) = \mathbf{I}\right) \\
&= M\sum_{k=0}^{M-1}\left|E_k\left(e^{jM\omega}\right)\right|^2 \\
&= M\sum_{k=0}^{M-1}1 \quad \left(\left|E_k\left(z^{j\omega}\right)\right| = 1\forall\omega\right) \\
&= M^2.
\end{aligned}
$$

Therefore, the analysis filters $H_k(z), k = 0, 1\cdots, (M-1)$ are power complementary.

(**Part b**) If $W = e^{-j\frac{2\pi}{M}}$, then

$$
H_k(z) = z^{-k}\sum_{i=0}^{M-1} z^{-i}W^{ik}E_i\left(z^M\right), \quad k = 1, 2, \cdots, M-1.
$$

Therefore,

$$
\begin{aligned}
H_k(z)\tilde{H}_k(z) &= \left(z^{-k}\mathbf{w}_k^T\mathbf{E}(z^M)\right)\left(z^k\left(\tilde{\mathbf{E}}(z^M)\right)^T\mathbf{w}_k^*\right) \\
&= \mathbf{w}_k^T\left(\mathbf{E}(z^M)\mathbf{E}^{\dagger}(z^M)\right)\mathbf{w}_k^*. \\
\implies \sum_{i=0}^{M-1}\tilde{H}_k\left(zW^i\right)H_k\left(zW^i\right) &= \sum_{i=0}^{M-1}\left(z^i\sum_{p=0}^{M-1}\left(zW^i\right)^pW^{-pk}\tilde{E}_p\left(z^MW^{iM}\right)\right)\left(z^{-i}\sum_{q=0}^{M-1}\left(zW^i\right)^{-q}W^{qk}E_q\left(z^MW^{iM}\right)\right) \\
&= \sum_{i=0}^{M-1}\left(\sum_{p=0}^{M-1}z^pW^{-p(k-i)}\tilde{E}_p\left(z^M\right)\right)\left(\sum_{q=0}^{M-1}z^{-q}W^{q(k+i)}E_q\left(z^M\right)\right) \\
&= \sum_{p=0}^{M-1}\sum_{q=0}^{M-1}\sum_{i=0}^{M-1}z^{p-q}W^{(q-p)k}W^{(p-q)i}\tilde{E}_p\left(z^M\right)E_q\left(z^M\right) \\
&= \sum_{p=0}^{M-1}\sum_{q=0}^{M-1}\left(z^{p-q}W^{(q-p)k}\tilde{E}_p\left(z^M\right)E_q\left(z^M\right)\sum_{i=0}^{M-1}W^{(p-q)i}\right) \quad \text{(Rearranging summations)} \\
&= \sum_{p=0}^{M-1}\sum_{q=0}^{M-1}\left(z^{p-q}W^{(q-p)k}\tilde{E}_p\left(z^M\right)E_q\left(z^M\right)M\delta(p-q)\right) \quad \left(\text{Using } \sum_{i=0}^{M-1}W^{ik} = M\delta(k)\right) \\
&= M\sum_{p=0}^{M-1}\tilde{E}_p\left(z^M\right)E_p\left(z^M\right) \quad \left(\text{Using } \delta(p-q) = \begin{cases}1, & p = q, \\ 0, & p \neq q.\end{cases}\right) \\
&= M\sum_{p=0}^{M-1}1 \quad \left(\text{On unit circle, } \tilde{E}_p\left(z^M\right)E_p\left(z^M\right) = \left|E_p\left(z^M\right)\right| = 1.\right) \\
\implies \sum_{i=0}^{M-1}\tilde{H}_k\left(zW^i\right)H_k\left(zW^i\right) &= M^2.
\end{aligned}
$$

Therefore, $H_k(z)\tilde{H}_k(z)$ satisfies M^{th} band property. Hence, each analysis filter is a spectral factor of an M^{th} band filter.

(**Part c**) The synthesis bank structure is shown in the Figure 1.

We use the following results proved in the proposition 1 at the end of this document.

(a) The system is alias free iff $E_k(z)R_k(z) = S(z), k = 0, 1, \cdots, M-1$ for some $S(z) \neq 0$.

(b) The overall system transfer function under this condition is $T(z) = z^{-(M-1)}S(z^M)$.

Since we need a stable synthesis bank and no information on the zeros of $E_k(z)$ are provided, we require $E_k(z)$ to be a factor of $S(z)\forall k$. Therefore, we choose $S(z) = \prod_{k=0}^{M-1}E_k(z)$. With this choice, the synthesis bank filters are

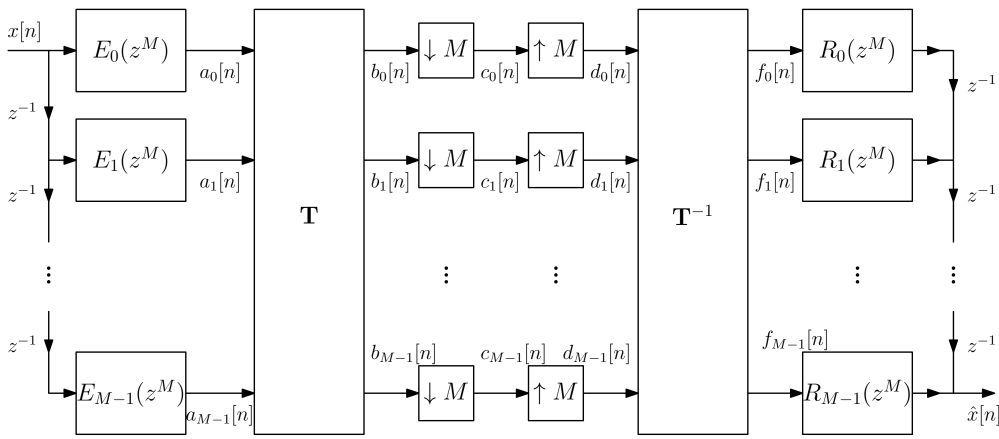

FIGURE 1. A generic analysis and synthesis filter bank system. Here $\mathbf{T} = \mathbf{W}^*$, where $\mathbf{W}$ is the $M \times M$ DFT matrix. The system is alias free if $E_k(z) R_k(z) = S(k), \quad k = 0, 1, \cdots, M-1$, giving the overall system transfer function as $T(z) = z^{-(M-1)} S\left(z^M\right)$.

given by

$$R_k(z) = \prod_{i \neq k} E_k(z), \quad k = 0, 1, \cdots, M-1.$$

The overall system response is

$$T(z) = z^{-(M-1)} \prod_{k=0}^{M-1} E_k\left(z^M\right).$$

The amplitude response is

$$\begin{aligned}
\left|T\left(e^{j\omega}\right)\right| &= \left| e^{-j\omega(M-1)} \prod_{k=0}^{M-1} E_k\left(e^{-j\omega M}\right) \right| \\
&= \left| e^{-j\omega(M-1)} \right| \prod_{k=0}^{M-1} \left| E_k\left(e^{-j\omega M}\right) \right| \\
\left|T\left(e^{j\omega}\right)\right| &= 1.
\end{aligned}$$

Hence, this choice of synthesis filter bank gives alias-free output without amplitude distortion.

Problem 3. (**5.13**)

$$\begin{aligned}
s(t) &= \sum_{n=-\infty}^{\infty} s_a(nT_1)\delta(t - nT_1) \\
&= \sum_{n=-\infty}^{\infty} s_a(t)\delta(t - nT_1) \quad \left(\text{Using } \delta(t) = \begin{cases} 1, & t = 0 \\ 0, & t \neq 0 \end{cases} \right) \\
&= s_a(t) \sum_{n=-\infty}^{\infty} \delta(t - nT_1) \\
&= s_a(t) \sum_{n=-\infty}^{\infty} e^{-j2\pi \frac{nt}{T_1}} \quad \left(\text{Using } \sum_{n=-\infty}^{\infty} \delta(t - nT_1) = \sum_{n=-\infty}^{\infty} e^{-j2\pi \frac{nt}{T_1}} \right) \\
&= \sum_{n=-\infty}^{\infty} s_a(t) e^{-j2\pi \frac{nt}{T_1}} \\
\Longrightarrow S(j\Omega) &= \int_{-\infty}^{\infty} s_a(t) e^{j\Omega t} dt
\end{aligned}$$

$$
\begin{aligned}
&= \int_{-\infty}^{\infty} \left(e^{j\Omega t} \sum_{n=-\infty}^{\infty} s_a(t) e^{-j2\pi \frac{nt}{T_1}} \right) \\
&= \sum_{n=-\infty}^{\infty} \left(\int_{-\infty}^{\infty} s_a(t) e^{j\left(\Omega - 2\pi \frac{nt}{T_1}\right)} \right) \quad \text{(Assume the exchange is possible.)} \\
S(j\Omega) &= \sum_{n=-\infty}^{\infty} S_a\left(j\Omega - j\frac{2\pi n}{T_1} \right).
\end{aligned}
$$

(**Part a**) We have

$$S_{a,k}(j\Omega) = X_a(j\Omega) H_{a,k}(j\Omega), \quad k = 0, 1.$$

$$
\begin{aligned}
\implies S_k(j\Omega) &= \sum_{n=-\infty}^{\infty} S_{a,k}\left(j\Omega - j\frac{2\pi n}{T_1} \right) \\
S_k(j\Omega) &= \sum_{n=-\infty}^{\infty} X_a\left(j\Omega - j\frac{2\pi n}{T_1} \right) H_{a,k}\left(j\Omega - j\frac{2\pi n}{T_1} \right), \quad k = 0, 1.
\end{aligned}
$$

$$
\begin{aligned}
\implies \hat{X}(j\Omega) &= F_{a,0}(j\Omega) S_0(j\Omega) + F_{a,1}(j\Omega) S_1(j\Omega) \\
&= \sum_{n=-\infty}^{\infty} X_a\left(j\Omega - j\frac{2\pi n}{T_1} \right) \left(F_{a,0}(j\Omega) H_{a,0}\left(j\Omega - j\frac{2\pi n}{T_1} \right) + F_{a,1}(j\Omega) H_{a,1}\left(j\Omega - j\frac{2\pi n}{T_1} \right) \right).
\end{aligned}
$$

(**Part b**) Given $T_1 = 2T$, $T = \frac{2\pi}{\Theta}$, $\Theta = 2\sigma$. Substituting $T_1 = \frac{2\pi}{\sigma}$, we have

$$\hat{X}(j\Omega) = \sum_{n=-\infty}^{\infty} X_a(j(\Omega - n\sigma)) \left(F_{a,0}(j\Omega) H_{a,0}(j(\Omega - n\sigma)) + F_{a,1}(j\Omega) H_{a,1}(j(\Omega - n\sigma)) \right). \tag{1}$$

Given that $X(j\Omega) = 0$ for $|\Omega| \geq \sigma$. Therefore, $X(j(\Omega - n\sigma)) \neq 0$ only if

$$
\begin{aligned}
-\sigma \quad &< \Omega - n\sigma < \quad \sigma \\
\implies \frac{\Omega}{\sigma} - 1 \quad &< n < \quad \frac{\Omega}{\sigma} + 1
\end{aligned}
$$

i.e, n can be an integer in an open interval of length 2. Hence, n can take at most two values. Therefore, only 2 terms can be non-zero for a given value of Ω.

(**Part c**) Given that $|F_{a,k}(j\Omega)| = 0$ for $|\Omega| \geq \sigma$. From (1), $\hat{X}(z) = 0$ for $|\Omega| \geq \sigma$.

Case $-\sigma < \Omega \leq 0$:

$n \in \left(\frac{\Omega}{\sigma} - 1, \frac{\Omega}{\sigma} + 1\right) \implies n \in (-2, 1) \implies n \in \{-1, 0\}$. In this case,

$$
\begin{aligned}
\hat{X}(j\Omega) &= X_a(j\Omega) \left(F_{a,0}(j\Omega) H_{a,0}(j\Omega) + F_{a,1}(j\Omega) H_{a,1}(j\Omega) \right) \\
&\quad + X_a(j\Omega + j\sigma) \left(F_{a,0}(j\Omega) H_{a,0}(j\Omega + j\sigma) + F_{a,1}(j\Omega) H_{a,1}(j\Omega + j\sigma) \right) \\
\hat{X}(j\Omega) &= \begin{bmatrix} X_a(j\Omega) & X_a(j\Omega + j\sigma) \end{bmatrix} \begin{bmatrix} H_{a,0}(j\Omega) & H_{a,1}(j\Omega) \\ H_{a,0}(j\Omega + j\sigma) & H_{a,1}(j\Omega + j\sigma) \end{bmatrix} \begin{bmatrix} F_{a,0}(j\Omega) \\ F_{a,1}(j\Omega) \end{bmatrix}.
\end{aligned}
$$

Therefore, the output is free from aliasing and distortion if

$$\begin{bmatrix} H_{a,0}(j\Omega) & H_{a,1}(j\Omega) \\ H_{a,0}(j\Omega + j\sigma) & H_{a,1}(j\Omega + j\sigma) \end{bmatrix} \begin{bmatrix} F_{a,0}(j\Omega) \\ F_{a,1}(j\Omega) \end{bmatrix} = \begin{bmatrix} T_1 \\ 0 \end{bmatrix}. \tag{2}$$

Case $0 < \Omega < \sigma$:

$n \in \left(\frac{\Omega}{\sigma} - 1, \frac{\Omega}{\sigma} + 1\right) \implies n \in (-1, 2) \implies n \in \{0, 1\}$. In this case,

$$
\begin{aligned}
\hat{X}(j\Omega) &= X_a(j\Omega) \left(F_{a,0}(j\Omega) H_{a,0}(j\Omega) + F_{a,1}(j\Omega) H_{a,1}(j\Omega) \right) \\
&\quad + X_a(j\Omega - j\sigma) \left(F_{a,0}(j\Omega) H_{a,0}(j\Omega - j\sigma) + F_{a,1}(j\Omega) H_{a,1}(j\Omega - j\sigma) \right) \\
\hat{X}(j\Omega) &= \begin{bmatrix} X_a(j\Omega) & X_a(j\Omega - j\sigma) \end{bmatrix} \begin{bmatrix} H_{a,0}(j\Omega) & H_{a,1}(j\Omega) \\ H_{a,0}(j\Omega - j\sigma) & H_{a,1}(j\Omega - j\sigma) \end{bmatrix} \begin{bmatrix} F_{a,0}(j\Omega) \\ F_{a,1}(j\Omega) \end{bmatrix}.
\end{aligned}
$$

Therefore, the output is free from aliasing and distortion if

$$\begin{bmatrix} H_{a,0}(j\Omega) & H_{a,1}(j\Omega) \\ H_{a,0}(j\Omega - j\sigma) & H_{a,1}(j\Omega - j\sigma) \end{bmatrix} \begin{bmatrix} F_{a,0}(j\Omega) \\ F_{a,1}(j\Omega) \end{bmatrix} = \begin{bmatrix} T_1 \\ 0 \end{bmatrix}. \tag{3}$$

(**Part d**) $H_{a,0}(j\Omega) = 1$ and $H_{a,1}(j\Omega) = j\Omega T_1$. Using $\sigma T_1 = 2\pi$, In this case the matrices in (2) and (3) are

$$\begin{bmatrix} 1 & j\Omega T_1 \\ 1 & j\Omega T_1 + j2\pi \end{bmatrix} \text{ and } \begin{bmatrix} 1 & j\Omega T_1 \\ 1 & j\Omega T_1 - j2\pi \end{bmatrix}.$$

Their determinants are $j2\pi$ and $-j2\pi$ respectively. Therefore, the matrices are non-singular.

Case $-\sigma < \Omega \leq 0$:

$$\begin{aligned}
\begin{bmatrix} F_{a,0}(j\Omega) \\ F_{a,1}(j\Omega) \end{bmatrix} &= \begin{bmatrix} 1 & j\Omega T_1 \\ 1 & j\Omega T_1 + j2\pi \end{bmatrix}^{-1} \begin{bmatrix} T_1 \\ 0 \end{bmatrix} \\
&= \frac{1}{j2\pi} \begin{bmatrix} j(\Omega T_1 + 2\pi) & -j\Omega T \\ -1 & 1 \end{bmatrix} \begin{bmatrix} T_1 \\ 0 \end{bmatrix} \\
\begin{bmatrix} F_{a,0}(j\Omega) \\ F_{a,1}(j\Omega) \end{bmatrix} &= \begin{bmatrix} \frac{(\Omega T_1 + 2\pi) T_1}{2\pi} \\ j\frac{2T_1}{2\pi} \end{bmatrix}
\end{aligned}$$

Case $0 \leq \Omega < \sigma$:

$$\begin{aligned}
\begin{bmatrix} F_{a,0}(j\Omega) \\ F_{a,1}(j\Omega) \end{bmatrix} &= \begin{bmatrix} 1 & j\Omega T_1 \\ 1 & j\Omega T_1 - j2\pi \end{bmatrix}^{-1} \begin{bmatrix} T_1 \\ 0 \end{bmatrix} \\
&= \frac{-1}{j2\pi} \begin{bmatrix} j(\Omega T_1 - 2\pi) & -j\Omega T \\ -1 & 1 \end{bmatrix} \begin{bmatrix} T_1 \\ 0 \end{bmatrix} \\
\begin{bmatrix} F_{a,0}(j\Omega) \\ F_{a,1}(j\Omega) \end{bmatrix} &= \begin{bmatrix} \frac{(2\pi - \Omega T_1) T_1}{2\pi} \\ -j\frac{2T_1}{2\pi} \end{bmatrix}.
\end{aligned}$$

Therefore, the synthesis filters are given by

$$F_{a,0}(j\Omega) = \begin{cases} \frac{2\pi}{\sigma}\left(1 - \frac{1}{\sigma}|\Omega|\right), & |\Omega| < \sigma \\ 0 & |\Omega| \geq \sigma \end{cases}$$

$$F_{a,1}(j\Omega) = \begin{cases} -j\frac{2}{\sigma}\text{sign}(\Omega), & |\Omega| < \sigma \\ 0 & |\Omega| \geq \sigma \end{cases}$$

Define ideal low-pass filter and the corresponding time-domain sinc$(\cdot)$ function as

$$\begin{aligned}
W(j\Omega) &= \begin{cases} 1 & |\Omega| < \frac{\sigma}{2} \\ 0 & |\Omega| \geq \frac{\sigma}{2} \end{cases}, \\
w(t) &= \frac{\sigma}{2\pi} \frac{\sin(\sigma t/2)}{(\sigma t/2)}.
\end{aligned}$$

We can write $F_{a,0}(j\Omega) = \frac{2\pi}{\sigma^2} W(j\Omega) * W(j\Omega)$. Therefore,

$$\begin{aligned}
f_{a,0}(t) &= \frac{2\pi}{\sigma^2} \times 2\pi (w_1(t))^2 \quad \left(\mathcal{F}^{-1}(X_1(j\Omega) * X_1(j\Omega)) = 2\pi x_1(t) x_2(t)\right) \\
\Longrightarrow f_{a,0}(t) &= 4\sin^2(\sigma t/2)/\sigma^2 t^2.
\end{aligned}$$

Similarly, $F_{a,0}(j\Omega) = -j\frac{2}{\sigma}\left(-W\left(j\left(\Omega + \frac{\sigma}{2}\right)\right) + W\left(j\left(\Omega - \frac{\sigma}{2}\right)\right)\right)$. Therefore,

$$\begin{aligned}
f_{a,1}(t) &= -j\frac{2}{\sigma}\left(-w(t) e^{-j\frac{\sigma t}{2}} + w(t) e^{j\frac{\sigma t}{2}}\right) \\
&= \frac{4}{\sigma} w(t) \sin(\sigma t/2) \\
f_{a,1}(t) &= \frac{1}{2\pi}\left(4\sin^2(\sigma t/2)/\sigma^2 t\right).
\end{aligned}$$

Proposition 1. *For the analysis and synthesis filter bank system in the Figure 1,*

(a) The overall system transfer function is independent of the non-singular matrix $\mathbf{T}$.

(b) The output of the system is alias-free iff $E_k(z) R_k(z) = S(z)$, $k = 0, 1, \cdots, M-1$ *for some* $S(z) \neq 0$.

(c) Under the alias-free condition, the overall system transfer function is $T(z) = z^{-(M-1)} S\left(z^M\right)$.

Proof. Let the intermediate signals in the system be as indicated in the Figure 1. Define

$$\begin{aligned}
\mathbf{A}(z) &= \begin{bmatrix} A_0(z) & A_1(z) & \cdots & A_{M-1}(z) \end{bmatrix}, \\
\mathbf{B}(z) &= \begin{bmatrix} B_0(z) & B_1(z) & \cdots & B_{M-1}(z) \end{bmatrix}, \\
\mathbf{C}(z) &= \begin{bmatrix} C_0(z) & C_1(z) & \cdots & C_{M-1}(z) \end{bmatrix}, \\
\mathbf{D}(z) &= \begin{bmatrix} D_0(z) & D_1(z) & \cdots & D_{M-1}(z) \end{bmatrix}, \\
\mathbf{F}(z) &= \begin{bmatrix} F_0(z) & F_1(z) & \cdots & F_{M-1}(z) \end{bmatrix}, \\
\mathbf{E}(z) &= \begin{bmatrix} E_0(z) & E_1(z) & \cdots & E_{M-1}(z) \end{bmatrix}, \\
\mathbf{R}(z) &= \begin{bmatrix} A_0(z) & A_1(z) & \cdots & A_{M-1}(z) \end{bmatrix} \\
\text{and } \mathbf{\Lambda}(z) &= \begin{bmatrix} 1 & 0 & 0 & & 0 \\ 0 & z^{-1} & 0 & \cdots & 0 \\ 0 & 0 & z^{-2} & \cdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & z^{-(M-1)} \end{bmatrix}.
\end{aligned}$$

Using this notation, we derive the output $\hat{X}(z)$ as the function of input $X(z)$ as follows:

$$\begin{aligned}
\mathbf{A}(z) &= X(z)\mathbf{\Lambda}(z)\mathbf{E}(z^M). && (4)\\
\mathbf{B}(z) &= \mathbf{T}\mathbf{A}(z) \\
\mathbf{B}(z) &= X(z)\mathbf{T}\mathbf{\Lambda}(z)\mathbf{E}(z^M). && (5)
\end{aligned}$$

$$\begin{aligned}
\mathbf{C}(z) &= \frac{1}{M}\sum_{i=0}^{M-1}\mathbf{B}\left(z^{\frac{1}{M}}W^i\right) \\
&= \frac{1}{M}\sum_{i=0}^{M-1}X\left(z^{\frac{1}{M}}W^i\right)\mathbf{T}\mathbf{\Lambda}\left(z^{\frac{1}{M}}W^i\right)\mathbf{E}(zW^{iM}) \\
&= \frac{1}{M}\sum_{i=0}^{M-1}X\left(z^{\frac{1}{M}}W^i\right)\mathbf{T}\mathbf{\Lambda}\left(z^{\frac{1}{M}}W^i\right)\mathbf{E}(z) \\
\mathbf{C}(z) &= \frac{1}{M}\mathbf{T}\left(\sum_{i=0}^{M-1}X\left(z^{\frac{1}{M}}W^i\right)\mathbf{\Lambda}\left(z^{\frac{1}{M}}W^i\right)\right)\mathbf{E}(z). && (6)
\end{aligned}$$

$$\begin{aligned}
\mathbf{D}(z) &= \mathbf{C}\left(z^M\right) \\
\mathbf{D}(z) &= \frac{1}{M}\mathbf{T}\left(\sum_{i=0}^{M-1}X\left(zW^i\right)\mathbf{\Lambda}\left(zW^i\right)\right)\mathbf{E}(z^M). && (7)
\end{aligned}$$

$$\begin{aligned}
\mathbf{F}(z) &= \mathbf{T}^{-1}\mathbf{D}(z) \\
\mathbf{F}(z) &= \frac{1}{M}\mathbf{T}\left(\sum_{i=0}^{M-1}X\left(zW^i\right)\mathbf{\Lambda}\left(zW^i\right)\right)\mathbf{E}(z^M). && (8)
\end{aligned}$$

We can write,

$$\begin{bmatrix} z^{-(M-1)}R_0\left(z^M\right) & z^{-(M-2)}R_1\left(z^M\right) & \cdots & R_1\left(z^M\right) \end{bmatrix} = \left(\mathbf{R}\left(z^M\right)\right)^T\left(z^{M-1}\mathbf{\Lambda}(z)\right)^{-1}.$$

Using this,

$$\begin{aligned}
\hat{X}(z) &= \left(\mathbf{R}\left(z^M\right)\right)^T z^{-(M-1)}\left(\mathbf{\Lambda}(z)\right)^{-1}\mathbf{F}(z) \\
&= \frac{z^{-(M-1)}}{M}\left(\mathbf{R}\left(z^M\right)\right)^T\left(\sum_{i=0}^{M-1}X\left(zW^i\right)\left(\mathbf{\Lambda}(z)\right)^{-1}\mathbf{\Lambda}\left(zW^i\right)\right)\mathbf{E}(z^M) \\
&= \frac{z^{-(M-1)}}{M}\left(\mathbf{R}\left(z^M\right)\right)^T\left(\sum_{i=0}^{M-1}X\left(zW^i\right)\mathbf{\Lambda}\left(W^i\right)\right)\mathbf{E}(z^M)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{z^{-(M-1)}}{M} \sum_{p=0}^{M-1} \sum_{i=0}^{M-1} R_p(z) X\left(zW^i\right) W^{-ip} E_p\left(z^M\right) \\
\hat{X}(z) &= \frac{z^{-(M-1)}}{M} \sum_{i=0}^{M-1} \left(X\left(zW^i\right) \sum_{p=0}^{M-1} R_p\left(z^M\right) E_p\left(z^M\right) W^{-ip} \right). \qquad (9)
\end{aligned}
$$

In matrix representation, the output can be written as

$$
\hat{X}(z) = \frac{z^{-(M-1)}}{M} \begin{bmatrix} X(z) & X(zW) & \cdots & X(W^{M-1}) \end{bmatrix} \mathbf{W} \begin{bmatrix} E_0\left(z^M\right) R_0\left(z^M\right) \\ E_1\left(z^M\right) R_1\left(z^M\right) \\ \vdots \\ E_{M-1}\left(z^M\right) R_{M-1}\left(z^M\right) \end{bmatrix}. \qquad (10)
$$

To obtain alias-free output, the necessary condition is

$$
\frac{1}{M} \mathbf{W} \begin{bmatrix} E_0\left(z^M\right) R_0\left(z^M\right) \\ E_1\left(z^M\right) R_1\left(z^M\right) \\ \vdots \\ E_{M-1}\left(z^M\right) R_{M-1}\left(z^M\right) \end{bmatrix} = \begin{bmatrix} S(z^M) \\ 0 \\ \vdots \\ 0 \end{bmatrix},
$$

for some $S(z) \neq 0$. Since $\mathbf{W}^* \mathbf{W} = M\mathbf{I}$, we have

$$
\begin{bmatrix} E_0\left(z^M\right) R_0\left(z^M\right) \\ E_1\left(z^M\right) R_1\left(z^M\right) \\ \vdots \\ E_{M-1}\left(z^M\right) R_{M-1}\left(z^M\right) \end{bmatrix} = \mathbf{W}^* \begin{bmatrix} S(z^M) \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} S(z^M) \\ S(z^M) \\ \vdots \\ S(z^M) \end{bmatrix} \qquad (11)
$$

$$
\text{i.e, } R_p(z) E_p(z) = S(z), \quad p = 0, 1, \cdots, M-1. \qquad (12)
$$

Under this condition, the output is

$$
\begin{aligned}
\hat{X}(z) &= z^{-(M-1)} \begin{bmatrix} X(z) & X(zW) & \cdots & X(W^{M-1}) \end{bmatrix} \begin{bmatrix} S(z^M) \\ 0 \\ \vdots \\ 0 \end{bmatrix} \\
\hat{X}(z) &= z^{-(M-1)} S\left(z^M\right) X(z). \qquad (13)
\end{aligned}
$$

Therefore, the transfer function is

$$
T(z) = \frac{\hat{X}(z)}{X(z)} = z^{-(M-1)} S\left(z^M\right). \qquad (14)
$$

□