INDIAN INSTITUTE OF SCIENCE
E9-252: MATHEMATICAL METHODS AND TECHNIQUES IN SIGNAL PROCESSING
HOME WORK #1 - SOLUTIONS, FALL 2014

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Problem 1. The noise-free signal is

$$y[n] = n^a u[n-1],$$

where a is an integer and $u[n]$ is unit step function. Let $Y(z)$ denote the z-transform of the signal. The signal is causal and hence the region of convergence of $Y(z)$ is $|z| \geq 1$.

Case $a \geq 0$:

The z-transform of unit step function is

$$\sum_{n=-\infty}^{\infty} u[n] z^{-n} = \frac{1}{1-z^{-1}} = \frac{z}{z-1}.$$

Differentiating w.r.t, z on both sides we have,

$$\begin{aligned} \sum_{n=-\infty}^{\infty} nu[n] z^{-n-1} &= \frac{1}{(z-1)^2}, \\ \Longrightarrow \sum_{n=-\infty}^{\infty} nu[n-1] z^{-n} &= \frac{z}{(z-1)^2}. \end{aligned}$$

Repeating the differentiation $a-1$ more times, we get

$$Y(z) = \sum_{n=-\infty}^{\infty} n^a u[n] z^{-n} = \frac{f(z)}{(z-1)^{a+1}},$$

for some polynomial $f(z)$.

Therefore $z=1$ is $(a+1)^{\text{th}}$ order pole i.e., 1 is $(a+1)^{\text{th}}$ order mode for the system.

Case $a \leq -2$:

We have

$$\begin{aligned} Y(z) &= \sum_{n=1}^{\infty} n^a z^{-n} \\ &\leq \sum_{n=1}^{\infty} n^a |z|^{-n} \\ &\leq \sum_{n=1}^{\infty} n^a \quad (|z| \geq 1) \\ &\leq \sum_{n=1}^{\infty} n^{-2} \quad (a \leq -2) \\ &= \frac{\pi}{6}. \end{aligned}$$

Therefore, $Y(z) < \infty \forall |z| \geq 1$. Hence there are no poles.

Case $a = -1$:

We have

$$\begin{aligned} Y(z) &= \sum_{n=1}^{\infty} \frac{1}{n} z^{-n}. \\ \Longrightarrow Y(1) &= \sum_{n=1}^{\infty} \frac{1}{n} = \infty. \end{aligned}$$

Therefore $z = 1$ is a pole.

Problem 2. 1.4.16 We have

$$\begin{aligned} \mathbf{x}[t+1] &= \mathbf{A}\mathbf{x}[t] + \mathbf{b}f[t], \\ y[t] &= \mathbf{c}^T\mathbf{x}[t] + df[t]. \end{aligned}$$

Taking z-Transform, we have

$$\begin{aligned} \mathbf{X}(z) &= (z\mathbf{I} - \mathbf{A})^{-1}\mathbf{b}F(z), \\ Y(z) &= \mathbf{c}^T\mathbf{X}(z) + dF(z). \end{aligned}$$

The system transfer function is given by

$$H(z) = \frac{Y(z)}{F(z)} = \mathbf{c}^T(z\mathbf{I} - \mathbf{A})^{-1}\mathbf{b} + d.$$

Similarly, for 1.22, we have the system transfer function given by,

$$\overline{H}(z) = \frac{Y(z)}{F(z)} = \overline{\mathbf{c}}^T(z\mathbf{I} - \overline{\mathbf{A}})^{-1}\overline{\mathbf{b}} + \overline{d}.$$

Substituting $\overline{\mathbf{A}} = \mathbf{T}^{-1}\mathbf{A}\mathbf{T}$, $\overline{\mathbf{b}} = \mathbf{T}^{-1}\mathbf{b}$, $\overline{\mathbf{c}} = \mathbf{T}^T\mathbf{c}$, $\overline{d} = d$ in this equation,

$$\begin{aligned} \overline{H}(z) &= \mathbf{c}^T\mathbf{T}\left(z\mathbf{I} - \mathbf{T}^{-1}\overline{\mathbf{A}}\mathbf{T}\right)^{-1}\mathbf{T}^{-1}\mathbf{b} + d, \\ &= \mathbf{c}^T\left(\mathbf{T}\left(z\mathbf{I} - \mathbf{T}^{-1}\overline{\mathbf{A}}\mathbf{T}\right)\mathbf{T}^{-1}\right)^{-1}\mathbf{b} + d \\ &= \overline{\mathbf{c}}^T(z\mathbf{I} - \overline{\mathbf{A}})^{-1}\overline{\mathbf{b}} + \overline{d} \\ \overline{H}(z) &= H(z). \end{aligned}$$

Therefore the system transfer functions are same. A and $\overline{A}$ have same eigen values.
1.4.17 (Part 1) From state-space equations, we have

$$\mathbf{x}[t+1] = \mathbf{A}\mathbf{x}[t] + \mathbf{b}f[t]. \tag{1}$$

To prove,

$$\mathbf{x}[t] = \mathbf{A}^t\mathbf{x}[0] + \sum_{k=0}^{t-1}\mathbf{A}^k\mathbf{b}f[t-1-k]. \tag{2}$$

Step 1: For $t = 1$, (2) true from the state-space equation (1).
Step 2: Assume true for $t = n$, i.e.,

$$\mathbf{x}[n] = \mathbf{A}^n\mathbf{x}[0] + \sum_{k=0}^{n-1}\mathbf{A}^k\mathbf{b}f[n-1-k].$$

We need to prove that the equation (2) is true for $t = n + 1$.

From state space equations, we have

$$\begin{aligned}
\mathbf{x}[n+1] &= \mathbf{A}\mathbf{x}[n] + \mathbf{b}f[n] \\
&= \mathbf{A}^{n+1}\mathbf{x}[0] + \mathbf{A}\sum_{k=0}^{n-1}\mathbf{A}^{k}\mathbf{b}f[n-1-k] + \mathbf{b}f[n] \quad (\text{use } l = k+1) \\
&= \mathbf{A}^{n+1}\mathbf{x}[0] + \sum_{l=1}^{(n+1)-1}\mathbf{A}^{l}\mathbf{b}f[(n+1)-1-l] + \mathbf{b}f[n] \\
&= \mathbf{A}^{n+1}\mathbf{x}[0] + \sum_{l=0}^{(n+1)-1}\mathbf{A}^{l}\mathbf{b}f[(n+1)-1-l].
\end{aligned}$$

i.e., the equation (2) is true for $t = n+1$.
This proves the equation (2) by induction.
(b) For time varying case,

$$\begin{aligned}
\mathbf{x}[t] &= \mathbf{A}[t-1]\mathbf{x}[t-1] + \mathbf{b}[t-1]f[t-1] \\
&= \mathbf{A}[t-1]\mathbf{A}[t-2]\mathbf{x}[t-2] + \mathbf{A}[t-1]\mathbf{b}[t-2]f[t-2] + \mathbf{b}[t-1]f[t-1] \\
&= \prod_{i=0}^{j}\mathbf{A}[t-1-i]\mathbf{x}[t-1-j] + \sum_{k=0}^{j}\left(\prod_{i=0}^{k}\mathbf{A}[t-1-i]\right)\mathbf{b}[t-1-k]f[t-1-k] \quad (j=1) \\
&\vdots \\
\mathbf{x}[t] &= \prod_{i=0}^{t-1}\mathbf{A}[t-1-i]\mathbf{x}[0] + \sum_{k=0}^{j}\left(\prod_{i=0}^{k}\mathbf{A}[t-1-i]\right)\mathbf{b}[t-1-k]f[t-1-k].
\end{aligned}$$

Problem 3. Note the following relations between ceil and floor functions,

$$c(x) = -f(-x), \tag{3}$$

$$c(x) - f(x) = \begin{cases} 0 & \text{if } x \text{ is an integer} \\ 1 & \text{otherwise.} \end{cases} \tag{4}$$

Since X is a continuous random variable,

$$Pr[X \text{ is an integer}] = 0 \tag{5}$$

Let μ_c and μ_f be the means of $c(x)$ and $f(x)$ respectively. Let σ_c^2 and σ_f^2 be the respective variances.
From (3),

$$\begin{aligned}
\mu_c &= -\mathbb{E}[f(-X)] \\
&= -\int_{-\infty}^{\infty} f(-x)\frac{1}{\sqrt{2\pi\sigma^2}}\exp\left\{-\frac{x^2}{2\sigma^2}\right\}dx \\
&= -\int_{-\infty}^{\infty} f(y)\frac{1}{\sqrt{2\pi\sigma^2}}\exp\left\{-\frac{y^2}{2\sigma^2}\right\}dy \quad (\text{using } y = -x) \\
\mu_c &= -\mu_f
\end{aligned} \tag{6}$$

From (4) and (5),

$$\begin{aligned}
\mu_c - \mu_f &= 0 \times Pr[X \text{ is an integer}] + 1 \times Pr[X \text{ is not an integer}] \\
\mu_c - \mu_f &= 1.
\end{aligned} \tag{7}$$

From (6) and (7),

$$\begin{aligned}
\mu_c &= 0.5, \\
\mu_f &= -0.5.
\end{aligned}$$

From (4) and (5),

$$\begin{aligned}\mathbb{E}\left[(c(X))^2\right] &= \mathbb{E}\left[(1+f(X))^2\right] \\ &= 1+2\mathbb{E}[f(X)]+\mathbb{E}\left[(f(X))^2\right] \\ \mathbb{E}\left[(c(X))^2\right] &= \mathbb{E}\left[(f(X))^2\right].\end{aligned}$$

Therefore, the variances are related as

$$\begin{aligned}\sigma_c^2 &= \mathbb{E}\left[(c(X))^2\right]-\mu_c^2 \\ &= \mathbb{E}\left[(f(X))^2\right]-\mu_f^2 \\ \sigma_c^2 &= \sigma_f^2.\end{aligned}$$

There is no closed form expression for the variance. The variance can be computed using the p.m.fs of the ceil and floor given by

$$\begin{aligned}Pr[c(X)=n] &= Pr[n-1<X\le n], \\ Pr[f(X)=n] &= Pr[n\le X<n+1].\end{aligned}$$

Following MATLAB code computes the mean and variances of ceil and floor functions. When $\sigma^2=1$, $\sigma_c^2=\sigma_f^2\approx 0.5834$.

```
x_range    = -100:100;
pmf_ceil   = 0.5*(erf(x_range) - erf(x_range - 1));
pmf_floor  = 0.5*(erf(x_range+1) - erf(x_range));
mean_ceil  = x_range*pmf_ceil'
mean_floor = x_range*pmf_floor'
var_ceil   = (x_range.^2)*pmf_ceil'  - mean_ceil^2
var_floor  = (x_range.^2)*pmf_floor' - mean_floor^2
```

Problem 4. 2.10.52 Sufficient to prove that $\mathcal{V}^\perp$ forms a closed group under addition and scalar multiplication. Since, for each $\underline{x}\in\mathcal{V}$, $\underline{x}\in\mathcal{S}$ also holds true, and hence the remaining properties will hold true.

(1) If $\underline{x},\underline{y}\in\mathcal{V}^\perp$, then $\langle\underline{x},\underline{v}\rangle=\langle\underline{y},\underline{v}\rangle=0\forall\underline{v}\in\mathcal{V}$. $\implies \langle\underline{x}+\underline{y},\underline{v}\rangle=0 \implies \underline{x}+\underline{y}\in\mathcal{V}^\perp$.

(2) $\langle\underline{0},\underline{v}\rangle=0\forall\mathcal{V}$. Therefore $\underline{0}\in\mathcal{V}^\perp$. $\underline{0}+\underline{x}=\underline{x}+\underline{x}=\underline{x}\forall\underline{x}\in\mathcal{V}^\perp$ is trivially satisfied since $\underline{x}\in\mathcal{S}$.

(3) Let $\underline{x}\in\mathcal{V}^\perp$. Since $\underline{x}\in\mathcal{S}$, $\exists\,\underline{y}\in\mathcal{S}$ such that $\underline{x}+\underline{y}=\underline{0}$. $\implies \langle\underline{y},\underline{v}\rangle=-\langle\underline{x},\underline{v}\rangle=0\forall\underline{v}\in\mathcal{V}^\perp$. Therefore $\underline{y}\in\mathcal{V}^\perp$.

(4) For $\underline{x},\underline{y},\underline{z}\in\mathcal{V}^\perp$, $\underline{x}+(\underline{y}+\underline{z})=(\underline{x}+\underline{y})+\underline{z}$ is trivially satisfied since $\underline{x},\underline{y},\underline{z}\in\mathcal{S}$.

(5) If $\underline{x}\in\mathcal{V}^\perp$ and a is a scalar, then $\langle a\underline{x},\underline{v}\rangle=a\langle\underline{x},\underline{v}\rangle=0\forall\underline{v}\in\mathcal{V}$. Therefore $a\underline{x}\in\mathcal{V}^\perp$

2.12.57 Sufficient to prove that $\mathcal{V}\cap\mathcal{W}$ forms a closed group under addition and scalar multiplication. Since, for each $\underline{x}\in\mathcal{V}\cap\mathcal{W}$, $\underline{x}\in\mathcal{S}$ also holds true, and hence the remaining properties will hold true.

(1) Let $\underline{x},\underline{y}\in\mathcal{V}\cap\mathcal{W}$. Then $\underline{x}+\underline{y}\in\mathcal{V}$ and $\underline{x}+\underline{y}\in\mathcal{W}$. Hence, $\underline{x}+\underline{y}\in\mathcal{V}\cap\mathcal{W}$.

(2) $\underline{0}\in\mathcal{V}$ and $\underline{0}\in\mathcal{W}$ and hence $\underline{0}\in\mathcal{V}\cap\mathcal{W}$.

(3) Let $\underline{x}\in\mathcal{V}\cap\mathcal{W}$. Its additive inverse $\underline{y}\in\mathcal{S}$ is unique. Since $\mathcal{V}$ and $\mathcal{W}$ are subspaces, $\underline{y}\in\mathcal{V}$ and $\underline{y}\in\mathcal{W}$. Therefore $\exists\,\underline{y}\in\mathcal{V}\cap\mathcal{W}$ such that $\underline{x}+\underline{y}=\underline{0}$.

(4) For $\underline{x},\underline{y},\underline{z}\in\mathcal{V}\cap\mathcal{W}$, $\underline{x}+(\underline{y}+\underline{z})=(\underline{x}+\underline{y})+\underline{z}$ is trivially satisfied since $\underline{x},\underline{y},\underline{z}\in\mathcal{S}$.

(5) Let $\underline{x}\in\mathcal{V}\cap\mathcal{W}$ and a be a scalar. Since $\mathcal{V}$ and $\mathcal{W}$ are subspaces, $a\underline{x}\in\mathcal{V}$ and $a\underline{x}\in\mathcal{W}$. Therefore $a\underline{x}\in\mathcal{V}\cap\mathcal{W}$.

2.12.63 Let $\mathcal{B}_v$ be an orthonormal basis for $\mathcal{V}$. Let $\mathcal{B}=\{\underline{v}\in\mathcal{B}_v \mid \exists\,\underline{w}\in\mathcal{W}\text{ such that }\langle\underline{v},\underline{w}\rangle\neq 0\}$.

Hence, we can chose an orthonormal basis $\mathcal{B}_w$ for $\mathcal{W}$ such that $\mathcal{B}\subset\mathcal{B}_w$.

Note that $\mathcal{B}$ forms orthonormal basis for $\mathcal{V}\cap\mathcal{W}$ and $\mathcal{B}_v\cup\mathcal{B}_w$ forms an orthonormal basis for $\mathcal{V}+\mathcal{W}$.

Therefore,

$$\begin{aligned}\dim(\mathcal{V}+\mathcal{W}) &= |\mathcal{V}+\mathcal{W}|, \\ &= |\mathcal{V}|+|\mathcal{W}|-|\mathcal{V}\cap\mathcal{W}|, \\ \dim(\mathcal{V}+\mathcal{W}) &= \dim(\mathcal{V})+\dim(\mathcal{W})-\dim(\mathcal{V}\cap\mathcal{W}),\end{aligned}$$

where $|\mathbf{A}|$ represents the number of elements in the set $\mathbf{A}$.

We have,

$$\mathcal{V} \oplus \mathcal{W} = (\mathcal{V} \oplus \{\underline{0}\}) + (\{\underline{0}\} \oplus \mathcal{V}),$$
$$\dim(\mathcal{V} \oplus \{\underline{0}\}) = \dim(\mathcal{V}),$$
$$\dim(\mathcal{W} \oplus \{\underline{0}\}) = \dim(\mathcal{W}),$$
$$(\mathcal{V} \oplus \{\underline{0}\}) \cap (\{\underline{0}\} \oplus \mathcal{V}) = \{\underline{0} \oplus \underline{0}\}.$$

Therefore,

$$\dim(\mathcal{V} \oplus \mathcal{W}) = \dim(\mathcal{V}) + \dim(\mathcal{W}).$$

Problem 5. (Part 1) Computing the inner products and norms of the signals, we have

$$\begin{aligned}
\langle f_1(t), f_2(t)\rangle &= \frac{5}{48}T, \\
\langle f_2(t), f_3(t)\rangle &= \frac{5}{48}T, \\
\langle f_1(t), f_3(t)\rangle &= 0,
\end{aligned}$$

$$\|f_1(t)\|^2 = \|f_2(t)\|^2 = \|f_3(t)\|^2 = \frac{T}{6}.$$

The distance between the vectors $\underline{a}$ and $\underline{b}$ is given by $\|\underline{a} - \underline{b}\| = \sqrt{\|\underline{a}\|^2 + \|\underline{b}\|^2 - 2\langle \underline{a}, \underline{b}\rangle}$. The distance between the signals are

$$\begin{aligned}
\|f_1(t) - f_2(t)\| &= \sqrt{\frac{T}{6} + \frac{T}{6} - \frac{5T}{24}} = \sqrt{\frac{T}{8}}, \\
\|f_2(t) - f_3(t)\| &= \sqrt{\frac{T}{6} + \frac{T}{6} - \frac{5T}{24}} = \sqrt{\frac{T}{8}}, \\
\|f_1(t) - f_3(t)\| &= \sqrt{\frac{T}{6} + \frac{T}{6} - 0} = \sqrt{\frac{T}{3}}.
\end{aligned}$$

The angle between vectors $\underline{a}$ and $\underline{b}$ is given by $\cos^{-1}\left(\frac{\langle \underline{a}, \underline{b}\rangle}{\|\underline{a}\|\|\underline{b}\|}\right)$.
Therefore, the angles between the signals are given by

$$\begin{aligned}
\theta_{12} &= \cos^{-1}\left(\frac{5}{8}\right), \\
\theta_{23} &= \cos^{-1}\left(\frac{5}{8}\right), \\
\theta_{13} &= \frac{\pi}{2}.
\end{aligned}$$

An orthonormal basis can be derived using Gram-Schmidt ortho-normalization procedure. Since $f_1(t)$ and $f_3(t)$ are orthogonal, normalizing them gives two of the basis. Let $v_1(t)$, $v_2(t)$ and $v_3(t)$ be orthonormal basis of the signal space.

$$\begin{aligned}
v_1(t) &= \frac{f_1(t)}{\|f_1(t)\|} = \sqrt{\frac{6}{T}} f_1(t), \\
v_2(t) &= \frac{f_3(t)}{\|f_3(t)\|} = \sqrt{\frac{6}{T}} f_3(t), \\
v_3(t) &= \frac{f_2(t) - \langle f_2(t), v_1(t)\rangle v_1(t) - \langle f_2(t), v_2(t)\rangle v_2(t)}{\|f_2(t) - \langle f_2(t), v_1(t)\rangle v_1(t) - \langle f_2(t), v_2(t)\rangle v_2(t)\|} \\
&= \frac{f_2(t) - \frac{5}{8} f_1(t) - \frac{5}{8} f_3(t)}{\left\| f_2(t) - \frac{5}{8} f_1(t) - \frac{5}{8} f_3(t) \right\|} \\
v_3(t) &= \sqrt{\frac{3}{7T}} \left(8 f_2(t) - 5 f_1(t) - 5 f_3(t)\right).
\end{aligned}$$

In the signal space with $v_1(t)$, $v_2(t)$ and $v_3(t)$ as the orthonormal basis, the signals $f_1(t)$, $f_2(t)$ and $f_3(t)$ are represented by the following vectors as shown in Figure 1.

$$\begin{aligned}
\underline{f}_1 &= \left(\sqrt{\frac{T}{6}}, 0, 0\right), \\
\underline{f}_2 &= \left(\frac{5}{48}\sqrt{6T}, \frac{5}{48}\sqrt{6T}, \frac{\sqrt{21T}}{24}\right), \\
\underline{f}_3 &= \left(0, \sqrt{\frac{T}{6}}, 0\right).
\end{aligned}$$

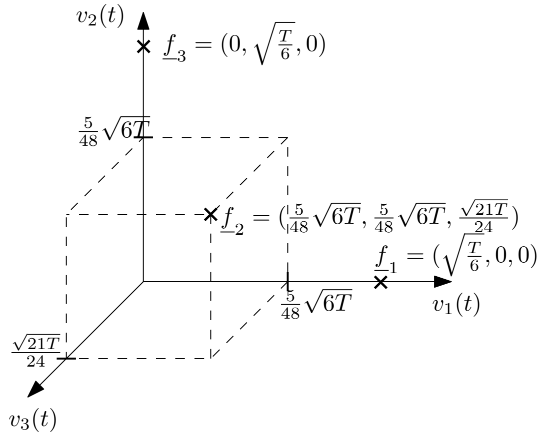

FIGURE 1. Signals $f_1(t)$, $f_2(t)$ and $f_3(t)$ are represented in signal space.

(Part 2) Received signal $r(t) = f_i(t) + \delta(t - t_i)$, where $t_i \sim U(Supp(f_i(t)))$. The noise signals are shown in Figure 2.

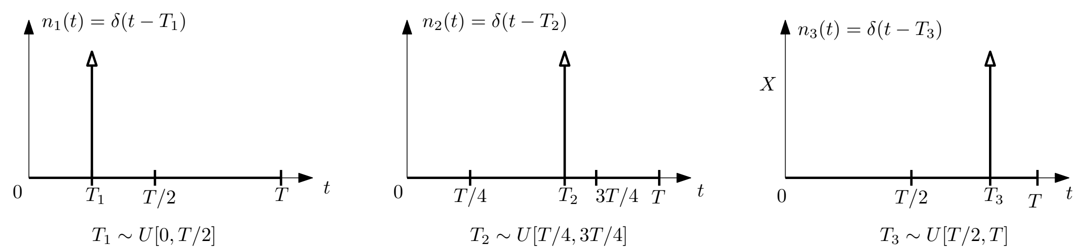

FIGURE 2. Noise signals corresponding to the input signals $f_1(t)$, $f_2(t)$ and $f_3(t)$ are shown. The noise is an impulse that occurs at a random time instant. This random time instant is uniformly distributed on the support of the corresponding signal.

We have,

$$\langle v_i(t), \delta(t - T)\rangle = v_i(T), \quad i = 1, 2, 3.$$

Therefore, the projection of $\delta(t - t_i)$ onto the vector space defined by the basis $v_1(t), v_2(t)$ and $v_3(t)$ is given by the parameterized curve

$$\{(v_1(t), v_2(t), v_3(t)) \mid t \in Supp(f_i(t))\}, \quad i = 1, 2, 3.$$

Hence, when the signal $s_i(t)$ is transmitted, the received signal lies on the parameterized curve given by

$$\mathbf{F}_i = \left\{\underline{f}_i + (v_1(t), v_2(t), v_3(t)) \mid t \in Supp(f_i(t))\right\}, \quad i = 1, 2, 3.$$

The received signal lies in the region given by $\mathbf{F}_1 \cup \mathbf{F}_2 \cup \mathbf{F}_3$. This is shown in Figure 3. Note that the three curves $\mathbf{F}_1$, $\mathbf{F}_2$ and $\mathbf{F}_3$ do not intersect.

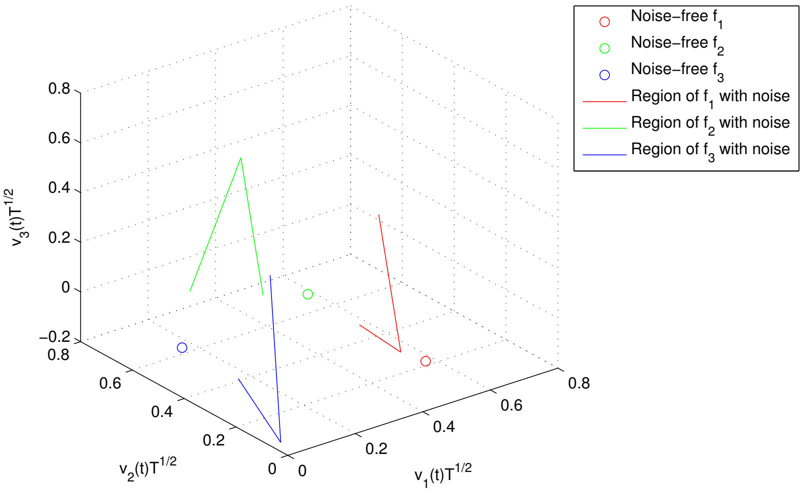

FIGURE 3. The signals $f_i(t)$ are shown. The regions $\mathbf{F}_i$ corresponding to received signal with noise when a signal $f_i(t)$, $i = 1, 2, 3$ is transmitted are also shown. The regions are V-shaped parametric curves and do not intersect.

We have the likelihood probability densities

$$p\left(\underline{r} \mid \underline{s}_i\right) \begin{cases} \neq 0, & \underline{r} \in \mathbf{F}_i \\ = 0, & \underline{r} \notin \mathbf{F}_i \end{cases}, \quad i = 1, 2, 3.$$

Therefore the aposterior probability densities are

$$p\left(\underline{f}_i \mid \underline{r}\right) = p\left(\underline{r} \mid \underline{f}_i\right) \frac{Pr\left[\underline{f}_i\right]}{p\left(\underline{r}\right)} \quad \begin{cases} \neq 0, & \underline{r} \in \mathbf{F}_i \\ = 0, & \underline{r} \notin \mathbf{F}_i \end{cases}, \quad i = 1, 2, 3.$$

Note that the regions $\mathbf{F}_1$, $\mathbf{F}_2$ and $\mathbf{F}_3$ do not intersect. Therefore, if $\underline{r} \in \mathbf{F}_i$, the aposterior probability is maximized for $\underline{f}_i$.

Therefore the optimal decision as are given by

$$\hat{f}(t) = \begin{cases} f_1(t), & \underline{r} \in \mathbf{R}_1 = \mathbf{F}_2^c \cap \mathbf{F}_3^c \\ f_2(t), & \underline{r} \in \mathbf{R}_1 = \mathbf{F}_2 \\ f_3(t), & \underline{r} \in \mathbf{R}_1 = \mathbf{F}_3. \end{cases}$$

The misclassification error is given by

$$\begin{aligned} P_e &= \sum_{i=1,2,3} Pr\left[f_i(t)\right] Pr\left[\hat{f}(t) \neq f_i(t) \mid f_i(t)\right] \\ &= \sum_{i=1,2,3} Pr\left[f_i(t)\right] Pr\left[\underline{r} \in \mathbf{R}_1^c \mid f_i(t)\right] \\ &= \sum_{i=1,2,3} Pr\left[f_i(t)\right] \times 0 \\ P_e &= 0. \end{aligned}$$

Remark: The orthonormal basis can be different depending on the order of signals used during Gram-Schmidt ortho-normalization.