(1) (a) The probability of correct decision is

$$
\begin{aligned}
P_c &= \sum_{i=1}^{k} \frac{1}{k} \operatorname{tr}\left(M_i^{\dagger} M_i \rho_i\right) \\
&\le \sum_{i=1}^{k} \frac{1}{k} \operatorname{tr}\left(M_i^{\dagger} M_i\right) \\
&= \frac{1}{k} \operatorname{tr}\left(\sum_{i=1}^{k} M_i^{\dagger} M_i\right) \\
&= \frac{1}{k} \operatorname{tr}(I) = \frac{d}{k}
\end{aligned}
$$

(b) Note from the 1st equality of (a)

$$
\begin{aligned}
P_c &= \frac{1}{k} \sum_{i=1}^{k} \operatorname{tr}\left(M_i^{\dagger} M_i \rho_i\right) \\
&\le \frac{1}{k} \sum_{i=1}^{k} \left\| M_i^{\dagger} M_i \right\|_1 \left\| \rho_i \right\|
\end{aligned}
$$

Choose $i^* = \arg\max_i \left\| \rho_i \right\|$

$$
\begin{aligned}
&\le \frac{1}{k} \cancel{\arg}\max_i \left\| \rho_i \right\| \underbrace{\sum_{i=1}^{k} \left\| M_i^{\dagger} M \right.}_{d} \\
&= \frac{d}{k} \max_i \left\| \rho_i \right\|
\end{aligned}
$$

$\boxtimes$

The strong subadditivity implies

$$H(C) + H(ABC) \leq H(AC) + H(BC)$$

Recall in terms of $E \cup F$, $E \cap F$, it is helpful.

(Subtracting $2H(C)$ from both sides)

$$H(ABC) - H(C) \leq H(AC) - H(C) + H(BC) - H(C)$$

$$H(AB|C) \leq H(A|C) + H(B|C)$$

(Conditional entropy defn)

$\boxtimes$

From the concavity of entropy

$$H(\gamma^A) \geq (1-\varepsilon) H(A)_\rho + \varepsilon H(A)_{\tilde{\rho}}$$

Having a classical variable X can only increase entropy.

$$H(A)_\gamma \leq H(AX)_\gamma$$

$$= H(A|X)_\gamma + H(X)$$

(From chain rule)

$$H(X) = H_2(\varepsilon)$$

$$H(A|X)_\gamma = (1-\varepsilon) H(A)_\rho + \varepsilon H(A)_{\tilde{\rho}}$$

$$\leq \varepsilon \left(H(A)_{\tilde{\rho}} - H(A)_\rho \right)$$

$$\leq 2\varepsilon \log(d_A)$$ (Similar proof done in class)

$$H(A|\gamma) \leq 2\varepsilon \log(d_A) + 2H(\varepsilon) \qquad (2)$$

Using (2) & (1), the result follows.

$\square$

Let us work out Part (A) similar to what we did in the class. This is Fannes ineq.

We know $\| \rho^A - \sigma^A \| \leq \varepsilon$.

Suppose $\| \rho^A - \sigma^A \| = \varepsilon$ (worst case)

Define the following density operators

$$\tilde{\rho}^A \triangleq \frac{1}{\varepsilon} \left| \rho^A - \sigma^A \right|$$

$$\tilde{\sigma}^A \triangleq \frac{1-\varepsilon}{\varepsilon} \left(\rho^A - \sigma^A \right) + \tilde{\rho}^A$$

$\tilde{\rho}^A$ is a valid density operator.

So is $\tilde{\sigma}^A$.

Construct a classical quantum state

$$\gamma^{XA} \triangleq (1-\varepsilon) |0\rangle\langle 0|^X \otimes \rho^A + \varepsilon |1\rangle\langle 1|^X \otimes \tilde{\rho}^A$$

Note that $\underbrace{\mathrm{tr}_X \left(\gamma^{XA} \right)}_{\gamma^A} = \underbrace{(1-\varepsilon) \rho^A + \varepsilon \tilde{\rho}^A}_{\text{Convex combination}}$

(2) Consider any two quantum states

(a) ρ^{AB} and σ^{AB}, $\|\rho^{AB} - \sigma^{AB}\|_1 \leq \varepsilon$.

From the monotonicity result under discarding of subsystems

$$\|\rho^{A} - \sigma^{A}\|_1 \leq \|\rho^{AB} - \sigma^{AB}\|_1$$

$$\leq \varepsilon \quad \text{(From problem statement)}$$

Consider $\left| I(A;B)_\rho - I(A;B)_\sigma \right|$

$$= \left| H(A)_\rho - H(A|B)_\rho - \left(H(A)_\sigma - H(A|B)_\sigma \right) \right| \quad \text{(Defn of mutual info.)}$$

$$\leq \underbrace{\left| H(A)_\rho - H(A)_\sigma \right|}_{\text{Part (A)}} + \underbrace{\left| H(A|B)_\rho - H(A|B)_\sigma \right|}_{\text{Part (B)}} \quad (\because \text{Triangle inequality})$$

From Alicki Fannes inequality done in class

$$\left| H(A|B)_\rho - H(A|B)_\sigma \right| \leq 4\varepsilon \log(d_A) + 2H_2(\varepsilon)$$

① This sorts out Part (B).

(2) (b) We are given an ensemble

$$\{ P_X(x), |\psi_x\rangle \}$$

$$\rho \triangleq \sum_x P_X(x) |\psi_x\rangle\langle\psi_x|$$

Let's start with a classical common randomness state $\sum_x P_X(x) |x\rangle\langle x|^X \otimes |x\rangle\langle x|^{X'}$ System B ——— ①

Apply a preparation map over ① to get

$$\sum_x P_X(x) |x\rangle\langle x|^X \otimes |\psi_x\rangle\langle\psi_x| \quad \text{System } B_1 \quad \text{——— ②}$$

From quantum data processing theorem

$$I(X;B) \geq I(X;B_1)$$

$$H(X) - H(X|B) \geq H(B_1) - H(B_1|X)$$

$$0 \geq H(\rho) - H(X)$$

shared randomness

$$H(X) \geq H(\rho)$$

$\Longrightarrow$ For orthogonal case eigen values of ρ to be $P_X(x)$ $\Rightarrow H(\rho) = H(X)$ happen

∎

3) a) In trying to ... coherent information, you must clearly "bring in your understanding that quantum correlations" are what the channel is trying to maximize.

Set up:

Consider a state $\phi^{AA_1'A_2'}$, input to a tensor product channel;

$$\mathcal{U}_{N_1}^{A_1' \to B_1E_1}\left(\phi^{AA_1'A_2'}\right) = \sigma^{AB_1E_1A_2'}$$

$$\mathcal{U}_{N_2}^{A_2' \to B_2E_2}\left(\phi^{AA_1'A_2'}\right) = \theta^{AB_2E_2A_1'}$$

$$\left.\begin{aligned}\mathcal{U}_{N_1} &\cong N_1^{A_1\to B_1E_1} \otimes I_{A_2'} \\ \mathcal{U}_{N_2} &= I_{A_1'} \otimes N_2^{A_2'\to B_2E_2}\end{aligned}\right\} \text{equivalent actions}$$

$\varphi \stackrel{\Delta}{=} \varphi^{AB_1E_1B_2E_2}$ (pure)

Consider $Q(N_1) + Q(N_2)$

$$= I(A > B_1A_2')_\sigma + I(A > B_2A_1')_\theta$$

$$= H(B_1)_\sigma - H(AA_2'B_1)_\sigma + H(B_2)_\theta - H(AA_1'B_2)_\theta$$

$$= H(B_1B_2)_\varphi - H(E_1)_\sigma - H(E_2)_\theta$$

$$= H(B_1B_2)_\varphi - H(AB_1B_2)_\varphi \quad (\because \text{product states})$$

$$\leq \max I(A > B_1B_2)_\varphi \text{ over choice of general form of states } \phi$$

(b) The Holevo information is concave in i/p distribution with fixed signaling states. —— (1)

It is convex in i/p distribution with fixed distribution —— (2)

Cases (1) or (2) are fine as long as one quantity is fixed.

If not, one must work out an alternating optimization program to solve the global optimization problem, which can be quite a task in higher dimensions