Quantum Information Theory
Homework-2 solutions

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1.

4.1.8

We have $\rho_j = \frac{1}{2}\left(\mathbb{1} + \vec{r}_j \cdot \vec{\sigma}\right)$

$$\rho = \sum_j p(j)\, \rho_j$$
$$= \frac{1}{2} \sum_j p(j)\left(\mathbb{1} + \vec{r}_j \cdot \vec{\sigma}\right)$$

Since $\sum_j p(j) = 1$,

$$\rho = \frac{1}{2}\left(\mathbb{1} + \sum_j p(j)\, \vec{r}_j \cdot \vec{\sigma}\right)$$

$$\rho = \frac{1}{2}\left(\mathbb{1} + \vec{r} \cdot \vec{\sigma}\right)$$

$$\therefore \quad \vec{r} = \sum_j p(j)\, \vec{r}_j$$

4.9.2.

$$\rho = \sum_x p(x) \rho_x$$

$$P_{success} = \sum_x p(x) \operatorname{Tr}(\rho_x \Lambda_x)$$

$$T \geq p(x) \rho_x \qquad \forall x.$$

$$P_{success} = \sum_x \operatorname{Tr}\left(p(x) \rho_x \Lambda_x\right)$$

$$P_{success} \leq \sum_x \operatorname{Tr}(T \Lambda_x)$$

$$\leq \operatorname{Tr}\left(T \sum_x \Lambda_x\right)$$

$$P_{success} \leq \operatorname{Tr}(T)$$

Now, for encoding n bits in a d-dimensional subspace,

$$p(x)\rho_x = \begin{cases} 2^{-n}\rho_x, & x \in \{0,1\}^n \\ 0 & \text{otherwise} \end{cases}$$

Since ρ_x has eigenvalues ≤ 1

$$2^{-n}\mathbb{1} \geq 2^{-n}\rho_x \quad \forall x.$$

$$\Rightarrow \quad P_{success} \leq \mathrm{Tr}\left(2^{-n}\mathbb{1}\right)$$

$$P_{success} \leq d\,2^{-n}$$

4.4.7

Consider

$$\rho' = \frac{1}{4}\rho + \frac{1}{4}X\rho X + \frac{1}{4}Y\rho Y + \frac{1}{4}Z\rho Z$$

Writing $\rho = \frac{1}{2}\left(\mathbb{1} + \sum_i r_i \sigma_i\right)$,

$$\sigma_j \rho \sigma_j = \frac{\sigma_j}{2}\left(\sigma_j + r_j \mathbb{1} + \sum_{i \neq j} r_i \sigma_i \sigma_j\right)$$

$$= \frac{1}{2}\left(\mathbb{1} + r_j \sigma_j - \sum_{i \neq j} r_i \sigma_i\right)$$

$$\Rightarrow \rho' = \frac{1}{4}\rho + \frac{1}{4}\left(\frac{3\mathbb{1}}{2} - \frac{r_x X + r_y Y + r_z Z}{2}\right)$$

$$= \frac{\mathbb{1}}{2} = \pi$$

Now, the channel

$$\rho \to (1-p)\rho + p\,\frac{I}{2}$$

can be written

$$\rho \to (1-p)\rho + \frac{p}{4}\left(\rho + X\rho X + Y\rho Y + Z\rho Z\right)$$

$$\rho \to \left(1-\frac{3p}{4}\right)\rho + \frac{p}{4}\left(X\rho X + Y\rho Y + Z\rho Z\right)$$

4.4.10

$$\rho = \begin{bmatrix} 1-p & \eta \\ \eta^* & p \end{bmatrix}$$

$$= (1-p)|0\rangle\langle 0| + \eta\,|0\rangle\langle 1| + \eta^*\,|1\rangle\langle 0| + p\,|1\rangle\langle 1|$$

Kraus operators for ADC:

$$A_0 = \sqrt{\gamma}\,|0\rangle\langle 1| \qquad A_1 = |0\rangle\langle 0| + \sqrt{1-\gamma}\,|1\rangle\langle 1|$$

$$A_0 \rho A_0^{\dagger} = \gamma |0\rangle\langle 1|\rho|1\rangle\langle 0|$$
$$= \gamma p\,|0\rangle\langle 0|$$

$$A_1 \rho A_1^{\dagger} = \left(|0\rangle\langle 0| + \sqrt{1-\gamma}\,|1\rangle\langle 1|\right)\rho\left(|0\rangle\langle 0| + \sqrt{1-\gamma}\,|1\rangle\langle 1|\right)$$
$$= (1-p)|0\rangle\langle 0| + \sqrt{1-\gamma}\,\eta\,|0\rangle\langle 1| + \sqrt{1-\gamma}\,\eta^*\,|1\rangle\langle 0|$$
$$+ (1-\gamma)p\,|1\rangle\langle 1|$$

$\therefore$ The ADC gives

$$\rho' = \begin{bmatrix} 1-(1-\gamma)p & \sqrt{1-\gamma}\,\eta \\ \sqrt{1-\gamma}\,\eta^* & (1-\gamma)p \end{bmatrix}$$

4.4.11

Using the result from 4.4.10 for a general ρ,

$$\rho = \begin{bmatrix} 1-p & \eta \\ \eta^* & p \end{bmatrix}$$

$$\mathcal{N}_1(\rho) = \begin{bmatrix} 1-(1-\gamma_1)p & \sqrt{1-\gamma_1}\,\eta \\ \sqrt{1-\gamma_1}\,\eta^* & (1-\gamma_1)p \end{bmatrix}$$

$$\mathcal{N}_2(\mathcal{N}_1(\rho)) = \begin{bmatrix} 1-(1-\gamma_2)(1-\gamma_1)p & \sqrt{1-\gamma_2}\sqrt{1-\gamma_1}\,\eta \\ \sqrt{1-\gamma_2}\sqrt{1-\gamma_1}\,\eta^* & (1-\gamma_2)(1-\gamma_1)p \end{bmatrix}$$

$$\therefore \mathcal{N}_2(\gamma_2)\circ\mathcal{N}_1(\gamma_1) = \mathcal{N}(\gamma_1+\gamma_2-\gamma_1\gamma_2)$$

4.4.13

$$\rho \to \langle k|\rho|k\rangle\, |k\rangle\langle k| \otimes \sigma_k$$

Consider the state

$$|\psi\rangle^{AB} = \sum_{ij} a_{ij} |ij\rangle$$

We have

$$\rho_{AB} = \sum_{ijkl} a_{ij} a_{kl} |ij\rangle\langle kl|$$

where $i, j, k, l \in \{0, 1\}$

$$\rho_B = Tr_A(\rho_{AB})$$

$$= \sum_{ijk} a_{ij} a_{ik} |j\rangle\langle k|$$

$$\mathcal{N}(\rho_B) = \sum_{ijkl} a_{ij} a_{ik} \langle l|j\rangle\langle k|l\rangle \, |l\rangle\langle l|$$

$$= \sum_{ij} a_{ij}^2 |j\rangle\langle j|$$

which is a separable state of the form $\sum_j c_j |j\rangle\langle j|$

with $c_j = \sum_i a_{ij}^2$

2.

(a) $\rho \xrightarrow{\mathcal{N}} (1-\varepsilon)\rho + \varepsilon |e\rangle\langle e|$

$\mathcal{N}: \mathcal{H}_2 \to \mathcal{H}_3 \qquad |e\rangle \perp |0\rangle, |1\rangle$

Consider

$$M_0 = \sqrt{1-\varepsilon}\left(|0\rangle_3\langle 0|_2 + |1\rangle_3\langle 1|_2\right)$$

$$M_1 = \sqrt{\varepsilon}\,|e\rangle_3\langle 0|_2$$

$$M_2 = \sqrt{\varepsilon}\,|e\rangle_3\langle 1|_2$$

$$M_0^\dagger M_0 = (1-\varepsilon)\,\mathbb{1}_2 \quad M_1^\dagger M_1 = \varepsilon|0\rangle\langle 0| \quad M_2^\dagger M_2 = \varepsilon|1\rangle\langle 1|$$

$$\sum M_i^\dagger M_i = \mathbb{1}$$

$$M_0 \rho M_0^\dagger = (1-\varepsilon)\left(|0\rangle_3\langle 0|_2 + |1\rangle_3\langle 1|_2\right)\rho\left(|0\rangle_2\langle 0|_3 + |1\rangle_2\langle 1|_3\right)$$

$$= (1-\varepsilon)\sum_{i,j=0,1} \rho_{ij}|i\rangle_3\langle j|_3$$

$$M_1 \rho M_1^\dagger = \varepsilon \rho_{00} |e\rangle_3 \langle e|_3$$

$$M_2 \rho M_2^\dagger = \varepsilon \rho_{11} |e\rangle_3 \langle e|_3$$

$$\sum_i M_i \rho M_i^\dagger = (1-\varepsilon)\rho + \varepsilon |e\rangle\langle e|$$

In $\mathcal{H}_3 \otimes \mathcal{H}_3$, ρ would be $\begin{pmatrix} \rho_{00} & \rho_{01} & 0 \\ \rho_{10} & \rho_{11} & 0 \\ 0 & 0 & 0 \end{pmatrix}$ but we can ignore!

(b) generalized dephasing channel

A dephasing channel only modifies the phases while keeping diagonal elements same:

$$\rho \to (1-p)\rho + p Z \rho Z$$

In a d-dimensional Hilbert space there are d phases

$$Z(z)|j\rangle = \exp(i 2\pi z j / d)|j\rangle \quad z \in \{0, \dots d-1\}$$

$\therefore$ The d-dimensional dephasing channel is

$$\rho \to \sum_{i=0}^{d-1} p_i Z(i) \rho Z(i)^\dagger$$

The Kraus operators are $\{Z(z)\}$ $z \in \{0, \dots d-1\}$

3.

8.21

(1) $H = \chi(a^\dagger b + b^\dagger a)$

$$\mathcal{U} = e^{-iH\Delta t}$$

Let us denote the state $|n\rangle_a |k\rangle_b$ as $|n, k\rangle$. The action of H, and therefore $\mathcal{U}$, on this should therefore preserve $n+k$.

$\therefore E_k = \langle k_b | \mathcal{U} | 0_b \rangle$ will be of the form

$$E_k = \sum_n \langle n-k, k | \mathcal{U} | n, 0 \rangle |n-k\rangle\langle n|$$

$$= \sum_n \langle 0,0 | \frac{a^{n-k}}{\sqrt{(n-k)!}} \frac{b^k}{\sqrt{k!}} \mathcal{U} \frac{a^{\dagger n}}{\sqrt{n!}} | 0, 0 \rangle |n-k\rangle\langle n|$$

$$= \sum_n \sqrt{\binom{n}{k}} \frac{1}{n!} \langle 0,0 | a^{n-k} b^k \mathcal{U} a^{\dagger n} \mathcal{U}^\dagger \mathcal{U} | 0,0 \rangle |n-k\rangle\langle n|$$

Since $\mathcal{U}$ preserves $n+k$, $\mathcal{U}|0,0\rangle = |0,0\rangle$.

$$E_k = \sum_n \sqrt{\binom{n}{k}} \frac{1}{n!} \langle 00| a^{n-k} b^k \mathcal{U} a^{\dagger n} \mathcal{U}^\dagger |00\rangle |n-k\rangle\langle n|$$

$$\mathcal{U} A^n \mathcal{U}^\dagger = \mathcal{U} A \mathcal{U}^\dagger \mathcal{U} A^{n-1} \mathcal{U} = (\mathcal{U} A \mathcal{U}^\dagger)^n$$

$$E_k = \sum_n \sqrt{\binom{n}{k}} \frac{1}{n!} \langle 00| a^{n-k} b^k (\mathcal{U} a^\dagger \mathcal{U}^\dagger)^n |00\rangle |n-k\rangle\langle n|$$

Now,

$$\mathcal{U} a^\dagger \mathcal{U}^\dagger = e^{-iH\Delta t} a^\dagger e^{iH\Delta t}$$

We have

$$e^X Y e^{-X} = \sum_{n=0}^{\infty} \frac{[(X)^n, Y]}{n!}$$

$$X = -iH\Delta t \qquad Y = a^\dagger$$

$$[X, Y] = -i\chi\Delta t\, b^\dagger$$

$$[X,[X,Y]] = -\chi^2\Delta t^2\, a^\dagger \ldots$$

$$\therefore\ \mathcal{U} a^\dagger \mathcal{U}^\dagger = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}(\chi\Delta t)^{2n} a^\dagger$$

$$- i\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}(\chi\Delta t)^{2n+1}\, b^\dagger$$

$$\mathcal{U} a^\dagger \mathcal{U}^\dagger = \cos(\chi\Delta t)\, a^\dagger - i\sin(\chi\Delta t)\, b^\dagger$$

$$\mathcal{U} a^{\dagger n} \mathcal{U}^\dagger = \left(a^\dagger\cos(\chi\Delta t) - b^\dagger i\sin(\chi\Delta t)\right)^n$$

$$= \sum_{r=0}^{n} (-i)^r \binom{n}{r} a^{\dagger n-r} b^{\dagger r} \cos^{n-r}(\chi\Delta t)\sin^r(\chi\Delta t)$$

Writing $\gamma = 1 - \cos^2(\chi\Delta t)$,

$$\mathcal{U} a^{\dagger n} \mathcal{U}^\dagger = \sum_{r=0}^{n} (-i)^r \binom{n}{r} a^{\dagger n-r} b^{\dagger r} \sqrt{(1-\gamma)^{n-r}\gamma^r}$$

Since the only non-zero terms of E_k will be those with equal applications of a & $a^\dagger$, and b & $b^\dagger$, only the $r=k$ terms will appear, giving

$$E_k = (-i)^r \sum_n \binom{n}{k}^{3/2} \frac{\sqrt{(1-\gamma)^{n-r}\gamma^r}}{n!} \langle 00| a^{n-k} b^k a^{\dagger n-k} b^{\dagger k} |00\rangle \, |n-k\rangle\langle n|$$

$$a^k a^{\dagger k}|0\rangle = \sqrt{k!}\, a^k |k\rangle = k!\,|0\rangle$$

We can discard the global phase & write

$$E_k = \sum_n \sqrt{\binom{n}{k}} \sqrt{(1-\gamma)^{n-k}\gamma^k}\, |n-k\rangle\langle n|$$

(2) $E_k = \langle k_b | U | 0_b \rangle$

$$E_k^\dagger E_k = \langle 0_b | U^\dagger | k_b \rangle \langle k_b | U | 0_b \rangle$$

$$\sum_k E_k^\dagger E_k = \langle 0_b | U^\dagger \Big(\sum_k |k_b\rangle\langle k_b|\Big) U | 0_b \rangle$$

$$= \langle 0_b | U^\dagger U | 0_b \rangle$$

$$\sum_k E_k^\dagger E_k = \mathbb{1}$$

$$\mathrm{Tr}\left(\sum_k \left(E_k \rho E_k^\dagger\right)\right) = \sum_k \mathrm{Tr}\left(E_k \rho E_k^\dagger\right)$$

$$= \sum_k \mathrm{Tr}\left(E_k^\dagger E_k \rho\right)$$

$$= \mathrm{Tr}\left[\left(\sum_k E_k^\dagger E_k\right)\rho\right]$$

$$\mathrm{Tr}\left(\sum_k \left(E_k \rho E_k^\dagger\right)\right) = \mathrm{Tr}(\rho)$$

8.23

$$|\psi\rangle = a|01\rangle + b|10\rangle$$

$$\rho = |a|^2 |01\rangle\langle 01| + |b|^2 |10\rangle\langle 10|$$
$$+ ab^* |01\rangle\langle 10| + a^* b |10\rangle\langle 01|$$

$$E_0 = |0\rangle\langle 0| + \sqrt{1-\gamma}\, |1\rangle\langle 1|$$

$$E_1 = \sqrt{\gamma}\, |0\rangle\langle 1|$$

$$(\mathcal{E}_{AD} \otimes \mathbb{1})\rho = \sum_{i=0,1} (E_i \otimes \mathbb{1})\rho(E_i^\dagger \otimes \mathbb{1})$$
$$= |a|^2 |01\rangle\langle 01| + (1-\gamma)|b|^2 |10\rangle\langle 10|$$
$$+ ab^* \sqrt{1-\gamma}\, |01\rangle\langle 10| + a^* b \sqrt{1-\gamma}\, |10\rangle\langle 01|$$
$$+ \gamma |b|^2 |00\rangle\langle 00|$$

$$(\mathcal{E}_{AD} \otimes \mathcal{E}_{AD})\rho = \sum_{i=0,1} (\mathbb{1} \otimes E_i)(\mathcal{E}_{AD} \otimes \mathbb{1})\rho(\mathbb{1} \otimes E_i^\dagger)$$

$$= (1-\gamma)\Big(|a|^2 |01\rangle\langle 01| + |b|^2 |10\rangle\langle 10|$$

$$+ a^* b |10\rangle\langle 01| + a b^* |01\rangle\langle 10|\Big) + \gamma |00\rangle\langle 00|$$

$$(\mathcal{E}_{AD} \otimes \mathcal{E}_{AD})\rho = (1-\gamma)\rho + \gamma |00\rangle\langle 00|$$

which is given by the Kraus operators

$$\tilde{E}_0^{dr} = \sqrt{1-\gamma}\ \mathbb{1}$$

$$\tilde{E}_1^{dr} = \sqrt{\gamma}\,\big[|00\rangle\langle 01| + |00\rangle\langle 10|\big]$$