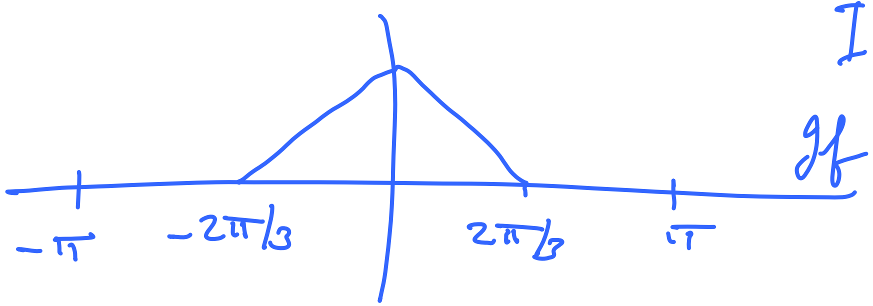
Fractional sampling rate alterations

$$x[n] \longrightarrow \boxed{\uparrow L} \xrightarrow{x_1(n)} \boxed{H(z)} \xrightarrow{x_2(n)} \boxed{\downarrow M} \longrightarrow y[n]$$

The effective sampling rate conversion is $\frac{L}{M}$.

Example:

Suppose we have a signal such that $X(e^{j\omega})$ is band limited to $|\omega| < 2\pi/3$



If we decimate by 2 $\longrightarrow$ "aliasing"

If we alter the rate by 1.5 $\longrightarrow$ "no aliasing" critically

$2\pi/3 \cdot 3/2 \longrightarrow \pi$

2 approaches for sampling rate conversion

$\frac{L}{M} = 1.5$ say

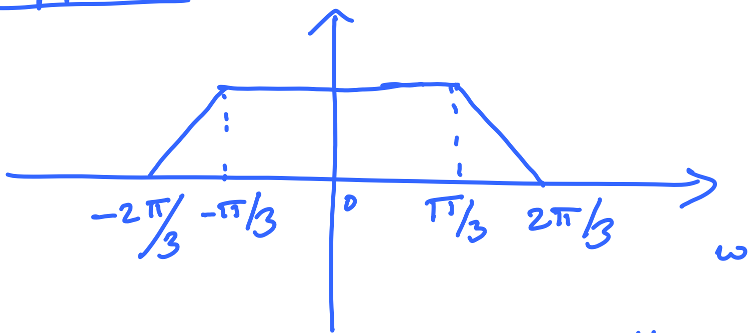
1) Signal $\xrightarrow{\text{Convert}}$ Analog $\longrightarrow$ Resample @ $f_s = 1.5 f_s'$
Discrete time

$\underbrace{\text{Discrete time} \to \text{Analog}}_{\text{D/A}}$ $\underbrace{\text{Analog} \to \text{Resample}}_{\text{Sampler}}$

" COSTLY "

2) Do a "judicious" use of 'downsampling' & 'upsampling'

Suppose $H(e^{j\omega})$



In general, we can reduce the sampling rate by any rational no M/L.

Quality of the filter $H(z)$ determines the quality of the o/p.

Stretched versions of $X(e^{j\omega L/M})$ govern the time domain meaning of L/M.

$L > M$ is very much possible

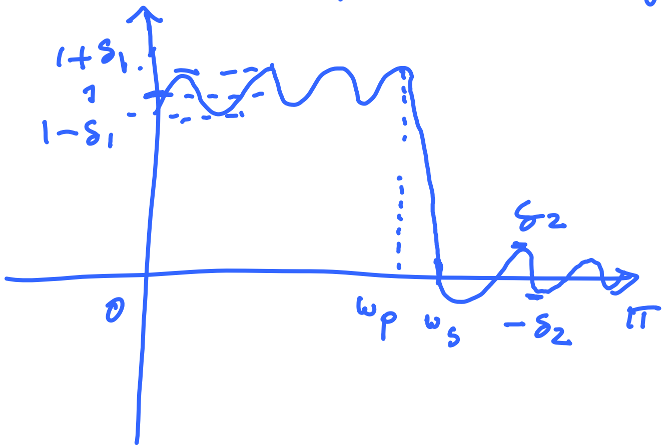
Focussing on the example

transition bandwidth $= \pi/3$

normalized tran. bw. $= \dfrac{\pi/3}{2\pi} = \dfrac{1}{6}$

(Δf)

For an equiripple design,



$$N \cong \left\{ \frac{2 \log_{10}\left(\frac{1}{10\,\delta_1 \delta_2}\right)}{3\,\Delta f} \right.$$

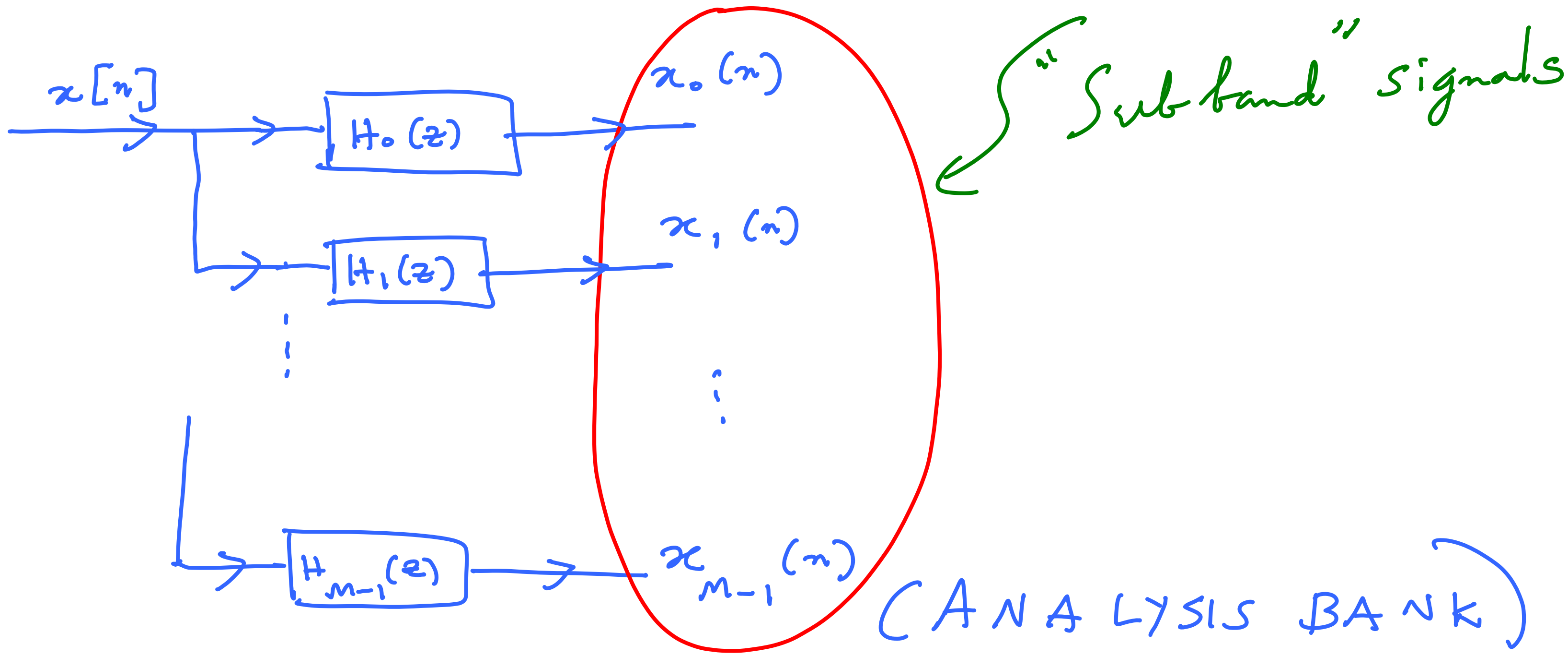
normalized transition bandwidth

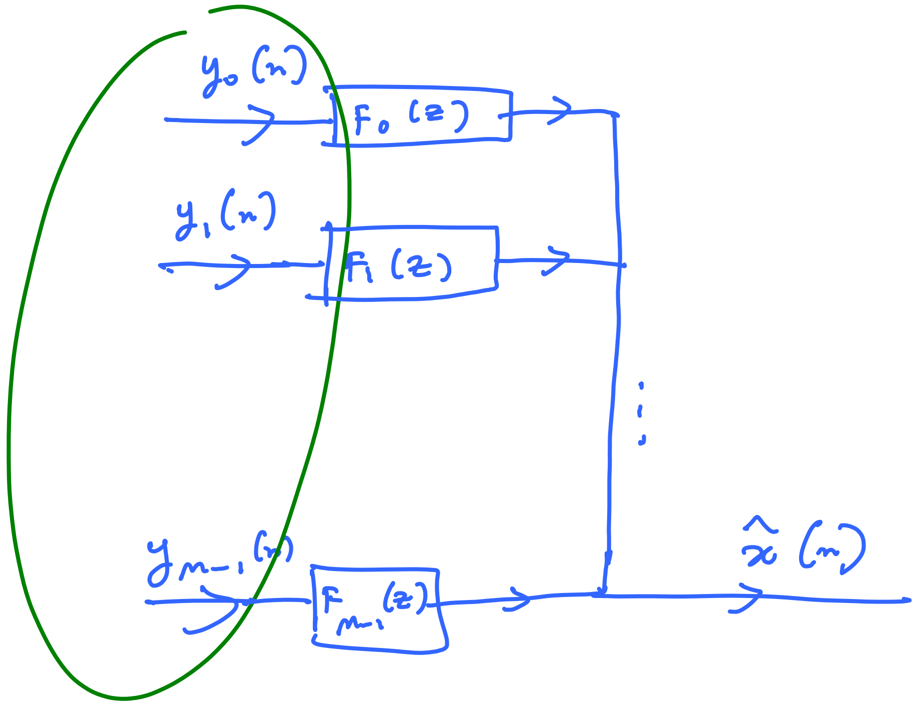
Suppose $\delta_1 = \delta_2 = 0.01$ for an equiripple filter, & $\Delta f = \frac{1}{6}$

$N \approx 11$ (11th order filter)

Digital Filter Banks

A digital filter bank is a collection of filters with a common i/p & a common o/p.

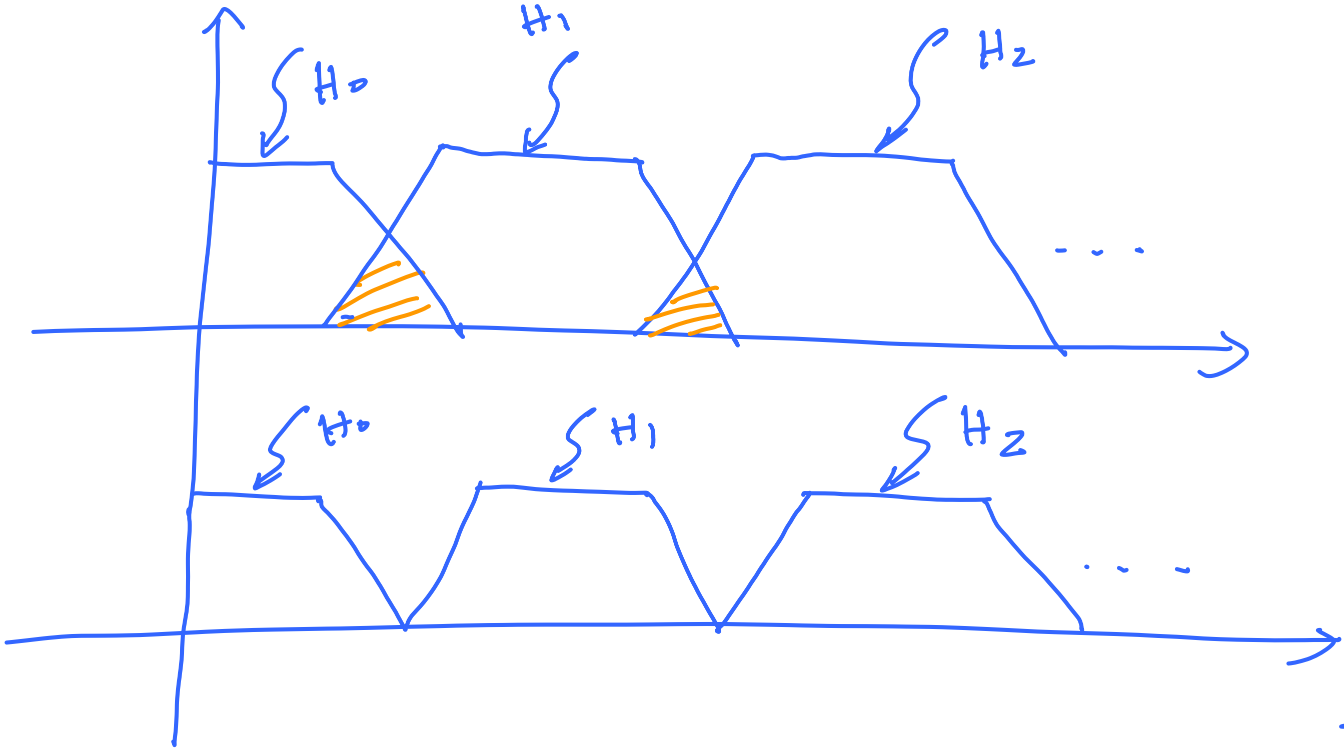




Sub band signals

"SYNTHESIS BANK"

What could be the nature of freq. responses for $H_i(z)$



"Overlapping"

more relaxed slope in the transition bandwidth region → Easy to design filters

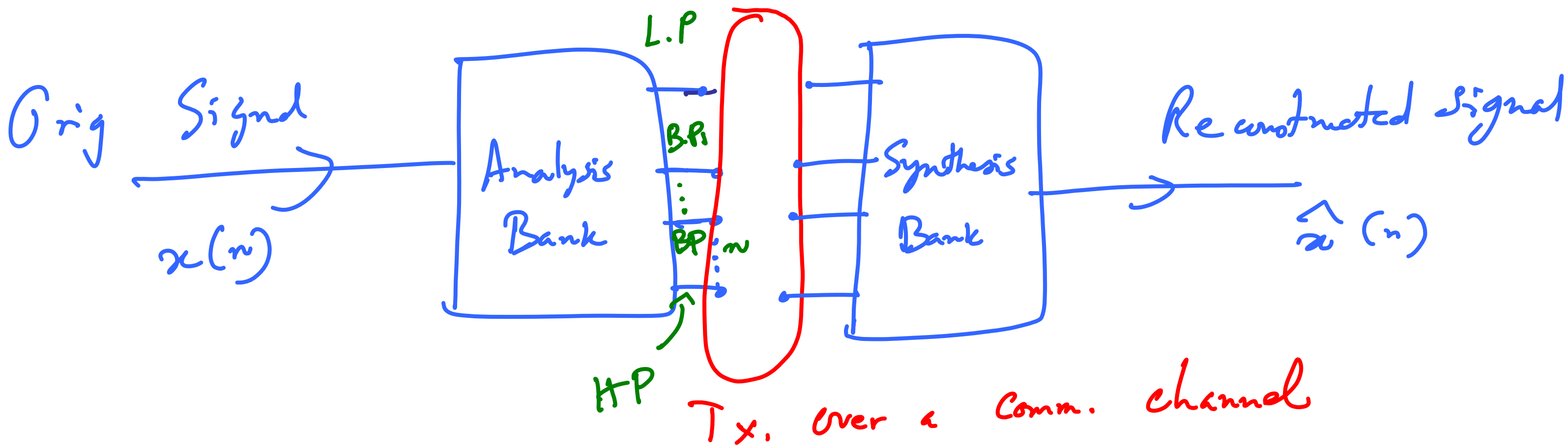
Aliasing is an issue

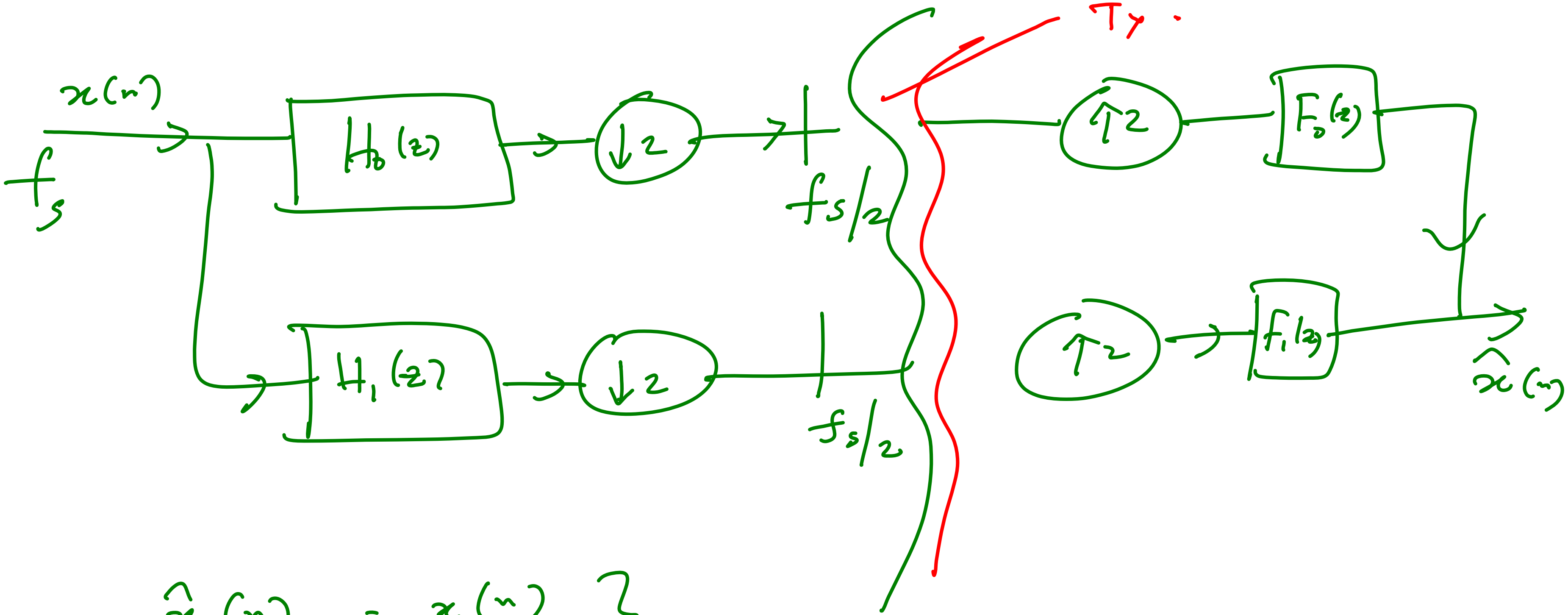
"Non overlapping"

Steeper freq. transition bandwidth slope

→ Filter orders could be prohibitive

→ "No aliasing"


Orig Signal
$x(n)$
Analysis Bank
L.P
B.P.
BP
HP
Synthesis Bank
Reconstructed signal
$\hat{x}(n)$
Tx. over a Comm. channel



Under what cond $\hat{x}(n) = x(n)$?

DFT as a simplest filter bank

The DFT matrix $(N \times N)$ is such that

$$W_N := \left[\, \omega_N^{km} \,\right] \qquad \omega_N = e^{-j 2\pi/N}$$

$$X(k) = \sum_{m=0}^{N-1} x(m)\, \omega_N^{km} \qquad \text{(DFT)}$$

$$x(m) = \frac{1}{N} \sum_{k=0}^{N-1} X(k)\, \omega_N^{-km} \qquad \text{(IDFT)}$$

The entry in the k^{th} row & m^{th} column is $e^{-j\frac{2\pi k m}{N}}$

DFT matrix

$$\underline{X} = W_N\, \underline{x}$$

time domain samples

freq. domain samples

ignore this subscript if you wish

Example:

$N = 2$

$$W_2 = \begin{bmatrix} e^{-j\frac{2\pi}{2}0\cdot 0} & e^{-j\frac{2\pi}{2}0\cdot 1} \\ e^{-j\frac{2\pi}{2}1\cdot 0} & e^{-j\frac{2\pi}{2}1\cdot 1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

low pass

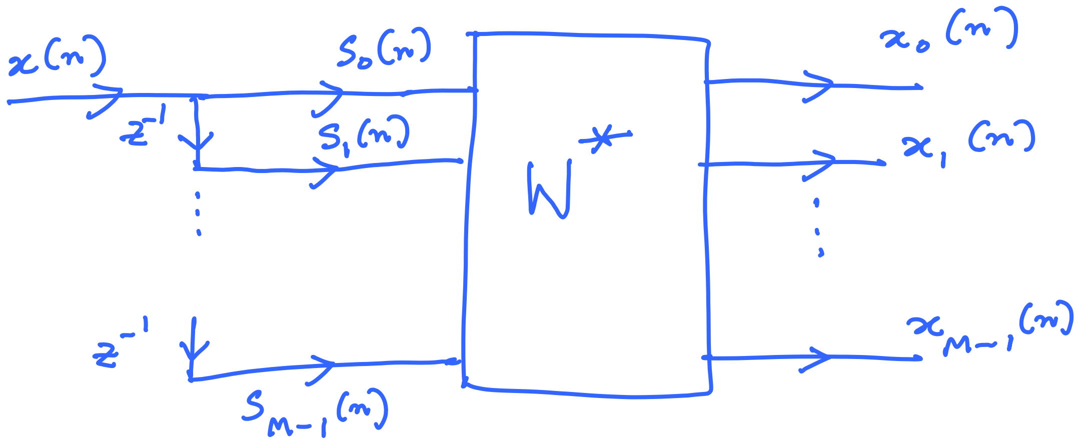
high pass

$1, 1 \longmapsto 1 + z^{-1}$ (LP)

$1, -1 \longmapsto 1 - z^{-1}$ (HP)

From the definition of W, we can interpret DFT as a filter bank

Consider the sequence $x(n)$ from which we generate M sequences $s_i(n)$; $i = 0, 1, \dots, M-1$ by passing $x(n)$ through a delay lines such that $s_i(n) = x(n-i)$



$$x_k(n) = \sum_{i=0}^{M-1} s_i(n)\, W_M^{-ki}$$

This is like the IDFT <u>without</u> a scale factor of $1/M$.

Taking Z- transforms,

$$X_k(z) = \sum_{i=0}^{M-1} S_i(z)\, W_M^{-ki}$$

But $S_i(z) = z^{-i} X(z)$

$$X_k(z) = \sum_{i=0}^{M-1} z^{-i} W_M^{-ki} X(z)$$

$$= \sum_{i=0}^{M-1} \left(z W_M^{k}\right)^{-i} X(z)$$

$$X_k(z) = H_k(z)\, X(z) \quad \text{where}$$

$$H_k(z) = H_0(z W_M^{k}) \;;\; H_0(z) = 1 + z^{-1} + z^{-2} + \dots + z^{-(M-1)}$$

modulation term

mother filter

The DFT system is like a filter bank with analysis filters $H_k(z)$

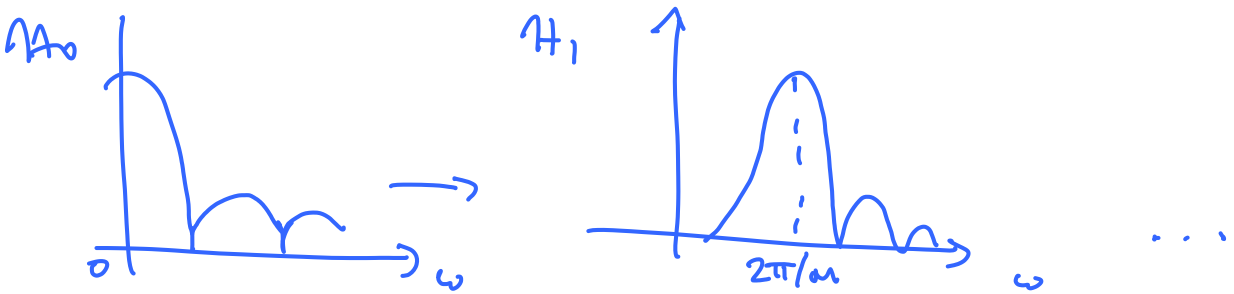
Exercise: Obtain the mag. freq response of $H_k(z)$ and plot them

HINT: Verify:

$$|H_0(z)| = \left| \frac{\sin(M\omega/2)}{\sin(\omega/2)} \right| \quad \text{--- } (1)$$

$$H_k(e^{j\omega}) = H_0\left(e^{j(\omega - 2\pi k/M)}\right) \quad \text{--- } (2)$$

Use (1) & (2) to sketch the freq. responses.



Time domain descriptions of multirate filters

$$y(n) = \begin{cases} \sum_{k=-\infty}^{\infty} x(k)\, h(nM-k) & M\text{-fold decimator} \\ \sum_{k=-\infty}^{\infty} x(k)\, h(n-kL) & L\text{ fold expander/inter} \\ \sum_{k=-\infty}^{\infty} x(k)\, h(nM-kL) & \frac{M}{L}\text{ fold decimator} \end{cases}$$

Interconnecting Systems

$$x(n) \longrightarrow \boxed{\downarrow M} \longrightarrow \boxed{\uparrow L} \longrightarrow y_1(n) \qquad \text{——} \qquad (a)$$

$$x(n) \longrightarrow \boxed{\uparrow L} \longrightarrow \boxed{\downarrow M} \longrightarrow y_2(n) \qquad \text{——} \qquad (b)$$

Question: Are (a) & (b) equivalent ?

Theorem: The systems (a) & (b) are equivalent if L and M are relatively prime. i.e., gcd (L, M) = 1

Sketch of proof:

$$Y_1(z) = \frac{1}{M} \sum_{k=0}^{M-1} X\Big(z^{L/M} \underbrace{\omega_M^{k}}\Big)$$

$$Y_2(z) = \frac{1}{M} \sum_{k=0}^{M-1} X\Big(z^{L/M} \underbrace{\omega_M^{kL}}\Big)$$

Claim: The set of numbers $S_1 = \{\omega_M^{k}\}$ distinct M^{th} roots of unity
(H.W.) & $S_2 = \{\omega_M^{kL}\}$ $\quad 0 \le k \le M-1$ is equal
iff gcd (L, M) = 1

Noble identities

$$x(n) \longrightarrow \boxed{\downarrow M} \longrightarrow \boxed{G(z)} \longrightarrow y_1(n) \quad \equiv \quad \longrightarrow \boxed{G(z^M)} \longrightarrow \boxed{\downarrow M} \longrightarrow y_2(n)$$

$$x(n) \longrightarrow \boxed{G(z)} \longrightarrow \boxed{\uparrow L} \longrightarrow y_3(n) \quad \equiv \quad \longrightarrow \boxed{\uparrow L} \longrightarrow \boxed{G(z^L)} \longrightarrow y_4(n)$$

Pathological Case : Suppose $G(z)$ is irrational i.e., $G(z) = z^{-1/2}$ (fractional delay filter)

If $x(n)$ is such that $x(2n) = 0$ $\quad M = 2$

$y_1(n)$ is not necessarily zero for all n ; $\quad y_2(n) = 0 \;\forall\, n$

Proof : (Non. irrational case)

(a) $$Y_2(z) = \frac{1}{M}\sum_{k=0}^{M-1} X\left(z^{\frac{1}{M}} w_M^{k}\right) G\left(\left(z^{\frac{1}{M}} w_M^{k}\right)^{M}\right)$$

$$= \frac{1}{M}\sum_{k=0}^{M-1} X\left(z^{\frac{1}{M}} w_M^{k}\right) G(z) \quad \text{——— } 1$$

$$Y_1(z) = \frac{1}{M}\sum_{k=0}^{M-1} X\left(z^{\frac{1}{M}} w^{k}\right) G(z) \quad \text{——— } 2$$

$$Y_1(z) = Y_2(z)$$

(b) IIIly $\quad Y_4(z) = G(z^L)\,X(z^L)$ —————— 3

$Y_3(z) = X(z)\,G(z) \rightarrow \boxed{\uparrow L} \rightarrow$

$= X(z^L)\,G(z^L)$ —————— 4

$Y_3(z) = Y_4(z)$

Polyphase Representation

Inventor: Bellanger 1976

Idea: Create a set of lower order filters that work efficiently under multirate operations

Example

Basic Idea: $H(z) = \sum_{n=-\infty}^{\infty} h(n) z^{-n}$

Separate $H(z)$ into even and odd coeffs.

$$H(z) = \sum_{n=-\infty}^{\infty} h(2n) z^{-2n} + z^{-1} \sum_{n=-\infty}^{\infty} h(2n+1) z^{-2n}$$

Let $E_0(z) = \sum_{n=-\infty}^{\infty} h(2n)\, z^{-n}$

$E_1(z) = \sum_{n=-\infty}^{\infty} h(2n+1)\, z^{-n}$

$H(z) = E_0(z^2) + z^{-1} E_1(z^2)$

Example : Suppose (a) FIR $H(z) = 1 + 2z^{-1} + 3z^{-2} + 4z^{-3}$ (FIR case)

$1 + 3z^{-2}$ $\quad$ $2z^{-1} + 4z^{-3}$

$E_0(z^2)$ $\quad$ $z^{-1}E_1(z^2)$

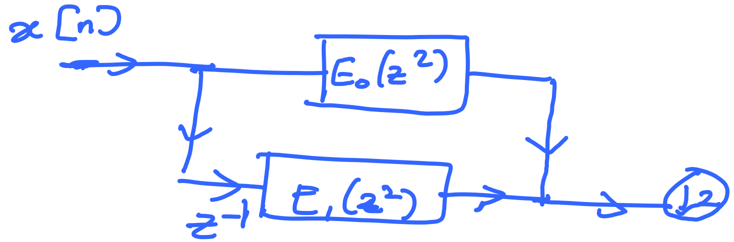
$E_0(z) = 1 + 3z^{-1}$ $\quad$ $E_1(z) = 2 + 4z^{-1}$

(b) I.I.R Case

$$H(z) = \frac{1}{1-\alpha z^{-1}} = \underbrace{\frac{1}{1-\alpha^2 z^{-2}}}_{E_0(z^2)} + \underbrace{\frac{\alpha z^{-1}}{1-\alpha^2 z^{-2}}}_{z^{-1}E_1(z^2)}$$

$$E_0(z) = \frac{1}{1-\alpha^2 z^{-1}}, \quad E_1(z) = \frac{\alpha}{1-\alpha^2 z^{-1}}$$

Having seen the basic trick, we would like to extend this idea for any integer M.



$$H(z) = \sum_{n=-\infty}^{\infty} h(nM)\, z^{-nM}$$

$$+ z^{-1} \sum_{n=-\infty}^{\infty} h(nM+1)\, z^{-nM}$$

$$\vdots$$

$$+ z^{-(M-1)} \sum_{n=-\infty}^{\infty} h(nM+M-1)\, z^{-nM}$$

This can be compactly written as

$$H(z) = \sum_{l=0}^{M-1} z^{-l} E_l(z^M) \qquad \text{(Type 1 poly phase)}$$

$$0 \le l \le M-1$$

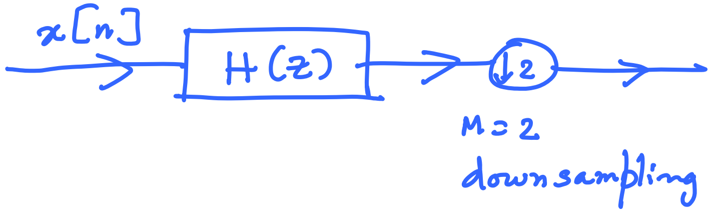
$$E_l(z) = \sum_{n=-\infty}^{\infty} e_l(n)\, z^{-n}; \quad e_l(n) = h(nM+l)$$

Exercise : Show that (Type 2 polyphase representation)

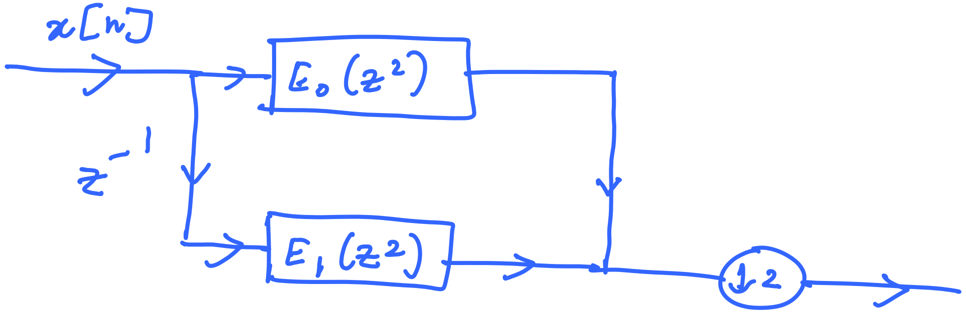
$$H(z) = \sum_{\ell=0}^{M-1} z^{-(M-1-\ell)} R_\ell(z^M)$$

where $R_\ell(z^M) = E_{M-1-\ell}(z)$ (Permutations of $E_\ell(z)$)

Efficient structures for decimation & interpolation filters



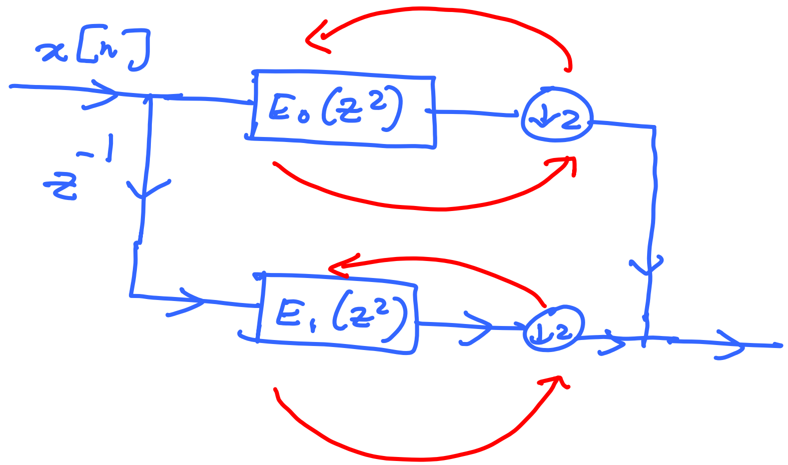
Let us do a Type 1 polyphase decomposition for H(z)



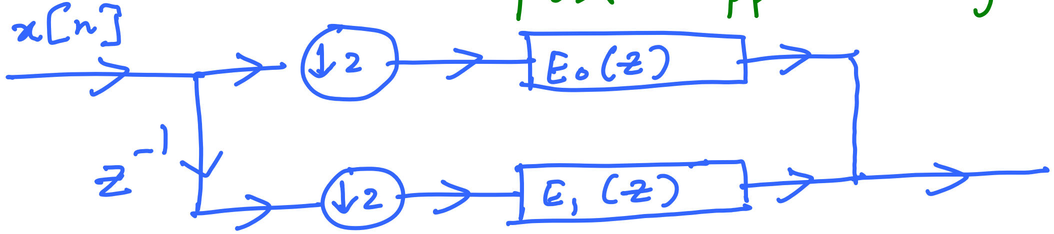
(a)

≡

Use Noble identities



Post application of Noble Identities



(b)

In the structure (a), because of downsampling, we consider only even numbered samples. During odd instants of time, the unit is just resting $\Rightarrow$ inefficient resource utilization

Consider structure (b): Suppose n_0 and n_1 are the orders of $E_0(z)$ and $E_1(z)$ / $N+1 = n_0 + n_1 + 2$

The multipliers & adders in each of $E_l(z)$ $l = 0, 1$ have 2 units of time for doing their job & continuously operative. (No resting time)

Interpolation filters

$$x(n) \rightarrow \boxed{\uparrow L} \rightarrow \boxed{H(z)} \rightarrow y(n)$$

$L = 2$

Clearly a direct implement of a filter post upsampling is inefficient because 'at least' 50% of the time ($L = 2$) we are filtering zeros.

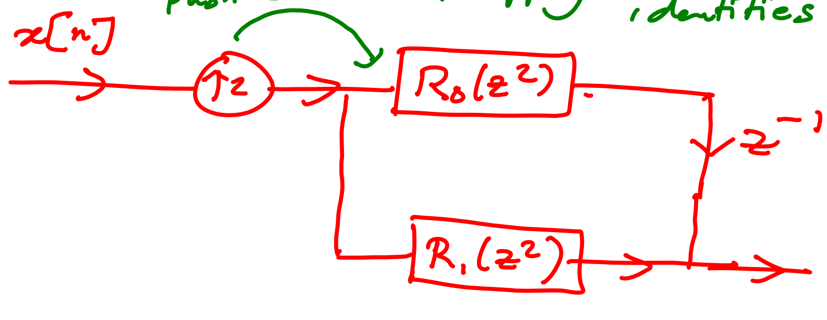
These multipliers "not resting" must complete the job in $\frac{1}{2}$ the time because o/p of the delay elements will change with time.

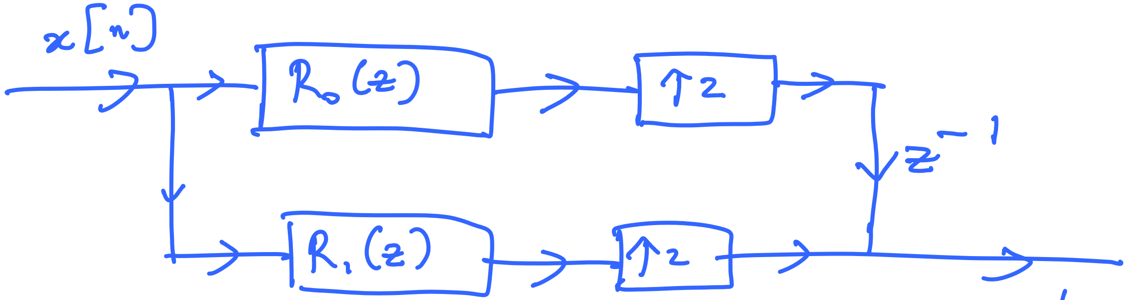
Using type 2 polyphase decomposition,

$$H(z) = R_1(z^2) + z^{-1} R_0(z^2)$$

$R_l(z)$ $l = 0, 1$ are operating at the i/p rate & no multiplier is resting

Push ↑2 inside + Apply Noble identities!





Efficient Structure for interpolation filters

Linear Phase FIR decimation filters

Let us suppose $H(z) = \sum_{n=0}^{N} h(n) z^{-n}$ such that $h(n) = h(N-n)$.

Let us investigate how symmetry in $h(n)$ reflects into the polyphase components.

Example: (a) Let $N = 4$ $\quad H(z) = 1 + 2z^{-1} + 4z^{-2} + 2z^{-3} + z^{-4}$

2 phase decomposition

$$E_0(z) = 1 + 4z^{-1} + z^{-2}$$
$$E_1(z) = 2 + 2z^{-1}$$

Each of the polyphase component filters are symmetric!

(b) Suppose $N = 5$

$$H(z) = 1 + 2z^{-1} + 4z^{-2} + 4z^{-3} + 2z^{-4} + z^{-5}$$

$$E_0(z) = 1 + 4z^{-1} + 2z^{-2}$$

$$E_1(z) = 2 + 4z^{-1} + z^{-2}$$

$E_0(\cdot)$ & $E_1(\cdot)$ are "mirrors" of each other!

Exercise: Devise an efficient architecture to exploit the mirror/symmetric properties of polyphase components in decimation & interpolation filters.